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SMARANDACHE MULTI-SPACE THEORY

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Preface to the Second Edition

Our WORLD is a multiple one both shown by the natural world and human beings. For example, the observation enables one knowing that there are infinite planets in the universe. Each of them revolves on its own axis and has its own seasons. In the human society, these rich or poor, big or small countries appear and each of them has its own system. All of these show that our WORLD is not in homogenous but in multiple. Besides, all things that one can acknowledge is determined by his eyes, or ears, or nose, or tongue, or body or passions, i.e., these six organs, which means the WORLD consists of *have* and *not have* parts for human beings. For thousands years, human being has never stopped his steps for exploring its behaviors of all kinds.

We are used to the idea that our space has three dimensions: *length, breadth* and *height* with time providing the fourth dimension of spacetime by Einstein. In the string or superstring theories, we encounter 10 dimensions. However, *we do not even know what the right degree of freedom is*, as Witten said. Today, we have known two heartening notions for sciences. One is the *Smarandache multi-space* came into being by purely logic. Another is the *mathematical combinatorics* motivated by a combinatorial speculation, i.e., *a mathematical science can be reconstructed from or made by combinatorialization*. Both of them contribute sciences for consistency of research with that human progress in 21st century.

What is a Smarandache multi-space? It is nothing but a union of n different spaces equipped with different structures for an integer $n \geq 2$, which can be used both for discrete or connected spaces, particularly for systems in nature or human beings. We think the *Smarandache multi-space* and the *mathematical combinatorics* are the best candidates for 21st century sciences. This is the reason that the author wrote this book in 2006, published by HEXIS in USA. Now 5 years have pasted after the first edition published. More and more results on Smarandache multi-spaces appeared. The purpose of this edition is to survey Smarandache multi-space theory including new published results, also show its applications to physics, economy and epidemiology.

There are 10 chapters with 71 research problems in this edition. Chapter 1 is a preparation for the following chapters. The materials, such as those of groups, rings, commutative rings, vector spaces, metric spaces and Smarandache multi-spaces including important results are introduced in this chapter.

Chapter 2 concentrates on multi-spaces determined by graphs. Topics, such as those of the valency sequence, the eccentricity value sequence, the semi-arc automorphism, the decomposition of graph, operations on graphs, hamiltonian graphs and Smarandache sequences on symmetric graphs with results are discussed in this chapter, which can be also viewed as an introduction to graphs and multi-sets underlying structures.

Algebraic multi-spaces are introduced in Chapter 3. Various algebraic multi-spaces, such as those of multi-systems, multi-groups, multi-rings, vector multi-spaces, multi-modules are discussed and elementary results are obtained in this chapter.

Chapters 4-5 continue the discussion of graph multi-spaces. Chapter 4 concentrates on voltage assignments by multi-groups and constructs multi-voltage graphs of type I, II with liftings. Chapter 5 introduces the multi-embeddings of graphs in spaces. Topics such as those of topological surfaces, graph embeddings in spaces, multi-surface embeddings, 2-cell embeddings, and particularly, combinatorial maps, manifold graphs with classification, graph phase spaces are included in this chapter.

Chapters 6-8 introduce Smarandache geometry, i.e., geometrical multi-spaces. Chapter 6 discusses the map geometry with or without boundary, including a short introduction on these paradoxist geometry, non-geometry, counter-projective geometry, anti-geometry with classification, constructs these Smarandache geometry by map geometry and finds curvature equations in map geometry. Chapter 7 considers these elements of geometry, such as those of points, lines, polygons, circles and line bundles in planar map geometry and Chapter 8 concentrates on pseudo-Euclidean geometry on R^n, including integral curves, stability of differential equations, pseudo-Euclidean geometry, differential pseudo-manifolds, $\cdots$, etc..

Chapter 9 discusses spacial combinatorics, which is the combinatorial counterpart of multi-space, also an approach for constructing Smarandache multi-spaces. Topics in this chapter includes the inherited graph in multi-space, algebraic multi-systems, such as those of multi-groups, multi-rings and vector multi-spaces underlying a graph, combinatorial Euclidean spaces, combinatorial manifolds, topological groups and topological multi-groups and combinatorial metric spaces. It should be noted that the topological group is a typical example of Smarandache multi-spaces in classical mathematics. The final chapter presents applications of Smarandache multi-spaces, particularly to physics, economy and epidemiology.

In fact, Smarandache multi-spaces underlying graphs are an important systematically notion for scientific research in 21st century. As a textbook, this book can be applicable for graduate students in combinatorics, topological graphs, Smarandache geometry, physics and macro-economy.

This edition was began to prepare in 2010. Many colleagues and friends of mine have given me enthusiastic support and endless helps in writing. Here I must mention some of them. On the first, I would like to give my sincerely thanks to Dr.Perze for his encourage and endless help. Without his encourage, I would do some else works, can not investigate Smarandache multi-spaces for years and finish this edition. Second, I would like to thank Professors Feng Tian, Yanpei Liu, Mingyao Xu, Jiyi Yan, Fuji Zhang and Wenpeng Zhang for them interested in my research works. Their encouraging and warmhearted supports advance this book. Thanks are also given to Professors Han Ren, Yuanqiu Huang, Junliang Cai, Rongxia Hao, Wenguang Zai, Goudong Liu, Weili He and Erling Wei for their kindly helps and often discussing problems in mathematics altogether. Partially research results of mine were reported at Chinese Academy of Mathematics & System Sciences, Beijing Jiaotong University, Beijing Normal University, East-China Normal University and Hunan Normal University in past years. Some of them were also reported at *The 2nd*, *3rd* and *7th Conference on Graph Theory and Combinatorics of China* in 2006, 2008 and 2011. My sincerely thanks are also give to these audiences discussing mathematical topics with me in these periods.

Of course, I am responsible for the correctness all of these materials presented here. Any suggestions for improving this book or solutions for open problems in this book are welcome.

L.F.Mao

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CHAPTER 1.

Preliminaries

What is a Smarandache multi-space? Why is it important to modern Science? A Smarandache multi-space $\widetilde{S}$ is a union of n different spaces $S_1,\ S_2,\ S_n$ equipped with some different structures, such as those of algebraic, topological, differential, $\cdots$ structures for an integer $n \geq 2$, introduced by Smarandache in 1969 [Sma2]. Whence, it is a systematic notion for developing modern sciences, not isolated but an unified way connected with other fields. Today, this notion is widely accepted by the scientific society. Applying it further will develops mathematical sciences in the 21st century, also enhances the ability of human beings understanding the WORLD. For introducing the readers knowing this notion, preliminaries, such as those of sets and neutrosophic sets, groups, rings, vector spaces and metric spaces were introduced in this chapter, which are more useful in the following chapters. The reader familiar with these topics can skips over this chapter.

§1.1 SETS

1.1.1 Set. A *set* Ξ is a category consisting of parts, i.e., a collection of objects possessing with a property $\mathscr{P}$, denoted usually by

$$\Xi = \{\ x \mid x \text{ possesses the property } \mathscr{P}\ \}.$$

If an element x possesses the property $\mathscr{P}$, we say that x is an element of the set Ξ, denoted by $x \in \Xi$. On the other hand, if an element y does not possesses the property $\mathscr{P}$, then we say it is not an element of Ξ and denoted by $y \notin \Xi$.

For examples,

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$
$$B = \{p \mid p \text{ is a prime number}\},$$
$$C = \{(x, y) \mid x^2 + y^2 = 1\},$$
$$D = \{\text{the cities in the World}\}$$

are 4 sets by definition.

Two sets Ξ_1 and Ξ_2 are said to be *identical* if and only if for $\forall x \in \Xi_1$, we have $x \in \Xi_2$ and for $\forall x \in \Xi_2$, we also have $x \in \Xi_1$. For example, the following two sets

$$E = \{1, 2, -2\} \text{ and } F = \{\ x \mid x^3 - x^2 - 4x + 4 = 0\}$$

are identical since we can solve the equation $x^3 - x^2 - 4x + 4 = 0$ and get the solutions $x = 1, 2$ or -2.

Let S, T be two sets. Define binary operations *union* $S \cup T$, *intersection* $S \cap T$ and S minus T respectively by

$$S \bigcup T = \{x \mid x \in S \text{ or } x \in T\}, \quad S \bigcap T = \{x \mid x \in S \text{ and } x \in T\}$$

and

$$S \setminus T = \{x \mid x \in S \text{ but } x \notin T\}.$$

Calculation shows that

$$A \bigcup E = \{1, 2, -2, 3, 4, 5, 6, 7, 8, 9, 10\},$$
$$A \bigcap E = \{1, 2\},$$
$$A \setminus E = \{3, 4, 5, 6, 7, 8, 9, 10\},$$
$$E \setminus A = \{-2\}.$$

The operations $\cup$ and $\cap$ possess the following properties.

Theorem 1.1.1 *Let X, T and R be sets. Then*

(i) $X \bigcup X = X$ *and* $X \bigcap X = X$;

(ii) $X \bigcup T = T \bigcup X$ *and* $X \bigcap T = T \bigcap X$;

(iii) $X \bigcup (T \bigcup R) = (X \bigcup T) \bigcup R$ *and* $X \bigcap (T \bigcap R) = (X \bigcap T) \bigcap R$;

(iv) $X \bigcup (T \bigcap R) = (X \bigcup T) \bigcap (X \bigcup R)$,

$X \bigcap (T \bigcup R) = (X \bigcap T) \bigcup (X \bigcap R)$.

Proof These laws (i)-(iii) can be verified immediately by definition. For the law (iv), let $x \in X \bigcup (T \bigcap R) = (X \bigcup T) \bigcap (X \bigcup R)$. Then $x \in X$ or $x \in T \bigcap R$, i.e., $x \in T$ and $x \in R$. Now if $x \in X$, we know that $x \in X \cup T$ and $x \in X \cup R$. Whence, we get that $x \in (X \bigcup T) \bigcap (X \bigcup R)$. Otherwise, $x \in T \bigcap R$, i.e., $x \in T$ and $x \in R$. We also get that $x \in (X \bigcup T) \bigcap (X \bigcup R)$.

Conversely, for $\forall x \in (X \bigcup T) \bigcap (X \bigcup R)$, we know that $x \in X \bigcup T$ and $x \in X \bigcup R$, i.e., $x \in X$ or $x \in T$ and $x \in R$. If $x \in X$, we get that $x \in X \bigcup (T \bigcap R)$. If $x \in T$ and $x \in R$, we also get that $x \in X \bigcup (T \bigcap R)$. Therefore, $X \bigcup (T \bigcap R) = (X \bigcup T) \bigcap (X \bigcup R)$ by definition.

Similarly, we can also get the law $\overline{X \cap T} = \overline{X} \cup \overline{T}$. □

Let Ξ_1 and Ξ_2 be two sets. If for $\forall x \in \Xi_1$, there must be $x \in \Xi_2$, then we say that Ξ_1 is a *subset* of Ξ_2, denoted by $\Xi_1 \subseteq \Xi_2$. A subset Ξ_1 of Ξ_2 is *proper*, denoted by $\Xi_1 \subset \Xi_2$ if there exists an element $y \in \Xi_2$ with $y \notin \Xi_1$ hold. It should be noted that the void (empty) set $\emptyset$ is a subset of all sets by definition. All subsets of a set Ξ naturally form a set $\mathscr{P}(\Xi)$, called the *power set* of Ξ.

For a set $X \in \mathscr{P}(\Xi)$, its complement $\overline{X}$ is defined by $\overline{X} = \{ y \mid y \in \Xi \text{ but } y \notin X\}$. Then we know the following result.

Theorem 1.1.2 *Let Ξ be a set, $S, T \subset \Xi$. Then*

(1) $X \cap \overline{X} = \emptyset$ and $X \cup \overline{X} = \Xi$;

(2) $\overline{\overline{X}} = X$;

(3) $\overline{X \cup T} = \overline{X} \cap \overline{T}$ and $\overline{X \cap T} = \overline{X} \cup \overline{T}$.

Proof The laws (1) and (2) can be immediately verified by definition. For (3), let

$x \in \overline{X \cup T}$. Then $x \in \Xi$ but $x \notin X \cup T$, i.e., $x \notin X$ and $x \notin T$. Whence, $x \in \overline{X}$ and $x \in \overline{T}$. Therefore, $x \in \overline{X} \cap \overline{T}$. Now for $\forall x \in \overline{X} \cap \overline{T}$, there must be $x \in \overline{X}$ and $x \in \overline{T}$, i.e., $x \in \Xi$ but $x \notin X$ and $x \notin T$. Hence, $x \notin X \cup T$. This fact implies that $x \in \overline{X \cup T}$. By definition, we find that $\overline{X \cup T} = \overline{X} \cap \overline{T}$. Similarly, we can also get the law $\overline{X \cap T} = \overline{X} \cup \overline{T}$. This completes the proof. □

For a set Ξ and $H \subseteq \Xi$, the set $\Xi \setminus H$ is said the *complement* of H in Ξ, denoted by $\overline{H}(\Xi)$. We also abbreviate it to $\overline{H}$ if each set considered in the situation is a subset of $\Xi = \Omega$, i.e., the *universal set*.

These operations on sets in $\mathscr{P}(\Xi)$ observe the following laws.

L1 Itempotent law. For $\forall S \subseteq \Omega$,

$$A \bigcup A = A \bigcap A = A.$$

L2 Commutative law. For $\forall U, V \subseteq \Omega$,

$$U \bigcup V = V \bigcup U; \ U \bigcap V = V \bigcap U.$$

L3 Associative law. For $\forall U, V, W \subseteq \Omega$,

$$U \bigcup \left(V \bigcup W\right) = \left(U \bigcup V\right) \bigcup W; \ U \bigcap \left(V \bigcap W\right) = \left(U \bigcap V\right) \bigcap W.$$

L4 Absorption law. For $\forall U, V \subseteq \Omega$,

$$U \bigcap \left(U \bigcup V\right) = U \bigcup \left(U \bigcap V\right) = U.$$

L5 Distributive law. For $\forall U, V, W \subseteq \Omega$,

$$U \bigcup \left(V \bigcap W\right) = \left(U \bigcup V\right) \bigcap \left(U \bigcup W\right); \ U \bigcap \left(V \bigcup W\right) = \left(U \bigcap V\right) \bigcup \left(U \bigcap W\right).$$

L6 Universal bound law. For $\forall U \subseteq \Omega$,

$$\emptyset \bigcap U = \emptyset, \emptyset \bigcup U = U; \ \Omega \bigcap U = U, \Omega \bigcup U = \Omega.$$

L7 Unary complement law. For $\forall U \subseteq \Omega$,

$$U \bigcap \overline{U} = \emptyset; \ U \bigcup \overline{U} = \Omega.$$

A set with two operations "$\bigcap$" and "$\bigcup$" satisfying the laws $L1 \sim L7$ is said to be a *Boolean algebra*. Whence, we get the following result.

Theorem 1.1.3 *For any set U, all its subsets form a Boolean algebra under the operations* "$\bigcap$" *and* "$\bigcup$".

1.1.2 Partially Order Set. Let Ξ be a set. The *Cartesian product* $\Xi \times \Xi$ is defined by

$$\Xi \times \Xi = \{(x, y)|\forall x, y \in \Xi\}$$

and a subset $R \subseteq \Xi \times \Xi$ is called a *binary relation* on Ξ. We usually write xRy to denote that $\forall(x, y) \in R$. A *partially order set* is a set Ξ with a binary relation $\preceq$ such that the following laws hold.

O1 Reflective Law. For $x \in \Xi$, xRx.

O2 Antisymmetry Law. For $x, y \in \Xi$, xRy and $yRx \Rightarrow x = y$.

O3 Transitive Law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

Denote by $(\Xi, \preceq)$ a partially order set Ξ with a binary relation $\preceq$. A partially ordered set with finite number of elements can be conveniently represented by a diagram in such a way that each element in the set Ξ is represented by a point so placed on the plane that point a is above another point b if and only if $b \prec a$. This kind of diagram is essentially a directed graph. In fact, a directed graph is correspondent with a partially set and vice versa. For example, a few partially order sets are shown in Fig.1.1 where each diagram represents a finite partially order set.

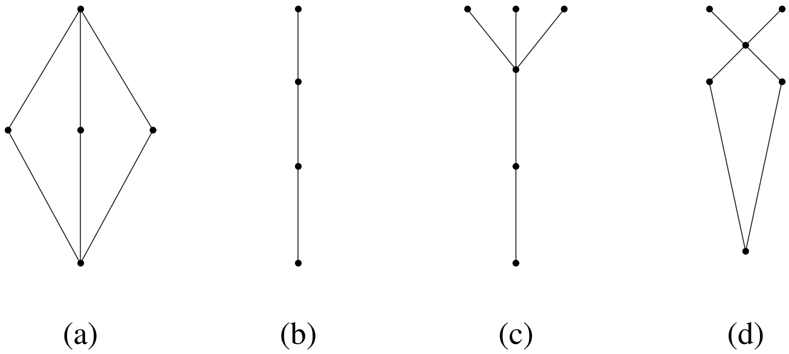

(a) (b) (c) (d)

Fig.1.1

An element a in a partially order set $(\Xi, \preceq)$ is called *maximal* (or *minimal*) if for $\forall x \in \Xi$, $a \preceq x \Rightarrow x = a$ (or $x \preceq a \Rightarrow x = a$). The following result is obtained by the definition of partially order sets and the induction principle.

Theorem 1.1.4 *Any finite non-empty partially order set* $(\Xi, \preceq)$ *has maximal and minimal elements.*

A partially order set $(\Xi, \preceq)$ is an *order set* if for any $\forall x, y \in \Xi$, there must be $x \preceq y$ or $y \preceq x$. For example, (b) in Fig.1.1 is such a ordered set. Obviously, any partially order set contains an order subset, which is easily find in Fig.1.1.

An *equivalence relation* $R \subseteq \Xi \times \Xi$ on a set Ξ is defined by

R1 Reflective Law. For $x \in \Xi$, xRx.

R2 Symmetry Law. For $x, y \in \Xi$, $xRy \Rightarrow yRx$

R3 Transitive Law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

Let R be an equivalence relation on set Ξ. We classify elements in Ξ by R with

$$R(x) = \{y | \ y \in \Xi \text{ and } yRx \ \}.$$

Then we get a useful result for the combinatorial enumeration following.

Theorem 1.1.5 *Let R be an equivalence relation on set* Ξ*. For* $\forall x, y \in \Xi$*, if there is an bijection* ς *between* $R(x)$ *and* $R(y)$*, then the number of equivalence classes under R is*

$$\frac{|\Xi|}{|R(x)|},$$

where x is a chosen element in Ξ*.*

Proof Notice that there is an bijection ς between $R(x)$ and $R(y)$ for $\forall x, y \in \Xi$. Whence, $|R(x)| = |R(y)|$. By definition, for $\forall x, y \in \Xi$, $R(x) \bigcap R(y) = \emptyset$ or $R(x) = R(y)$. Let T be a representation set of equivalence classes, i.e., choice one element in each class. Then we get that

$$|\Xi| = \sum_{x \in T} |R(x)| = |T||R(x)|.$$

Whence, we know that

$$|T| \ = \ \frac{|\Xi|}{|R(x)|}. \qquad \square$$

1.1.3 Neutrosophic Set. Let $[0, 1]$ be a closed interval. For three subsets $T, I, F \subseteq [0, 1]$ and $S \subseteq \Omega$, define a relation of element $x \in \Omega$ with the subset S to be $x(T, I, F)$, i.e., the *confidence set* T, the *indefinite set* I and the *fail set* F for $x \in S$. A set S with three subsets T, I, F is said to be a *neutrosophic set*. We clarify the conception of neutrosophic set by set theory following.

Let Ξ be a set and $A_1, A_2, \cdots, A_k \subseteq \Xi$. Define $3k$ functions $f_1^x, f_2^x, \cdots, f_k^x$ by $f_i^x : A_i \to [0,1]$, $1 \le i \le k$, where $x = T, I, F$. Denoted by $(A_i; f_i^T, f_i^I, f_i^F)$ the subset A_i with three functions f_i^T, f_i^I, f_i^F, $1 \le i \le k$. Then

$$\bigcup_{i=1}^{k}\left(A_i; f_i^T, f_i^I, f_i^F\right)$$

is a union of neutrosophic sets. Some extremal cases for this union is in the following, which convince us that neutrosophic sets are a generalization of classical sets.

Case 1 $f_i^T = 1$, $f_i^I = f_i^F = 0$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k}\left(A_i; f_i^T, f_i^I, f_i^F\right) = \bigcup_{i=1}^{k} A_i.$$

Case 2 $f_i^T = f_i^I = 0$, $f_i^F = 1$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k}\left(A_i; f_i^T, f_i^I, f_i^F\right) = \overline{\left(\bigcup_{i=1}^{k} A_i\right)}.$$

Case 3 There is an integer s such that $f_i^T = 1$ $f_i^I = f_i^F = 0$, $1 \le i \le s$ but $f_j^T = f_j^I = 0, f_j^F = 1$ for $s+1 \le j \le k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i, f_i) = \bigcup_{i=1}^{s} A_i \bigcup \overline{\left(\bigcup_{i=s+1}^{k} A_i\right)}.$$

Case 4 There is an integer l such that $f_l^T \ne 1$ or $f_l^F \ne 1$.

In this case, the union is a general neutrosophic set. It can not be represented by abstract sets. If $A \bigcap B = \emptyset$, define the function value of a function f on the union set $A \bigcup B$ to be

$$f\left(A \bigcup B\right) = f(A) + f(B)$$

and

$$f\left(A \bigcap B\right) = f(A)f(B).$$

Then if $A\bigcap B \neq \emptyset$, we get that

$$f\left(A\bigcup B\right) = f(A) + f(B) - f(A)f(B).$$

Generally, we get the following formulae.

$$f\left(\bigcap_{i=1}^{l} A_i\right) = \prod_{i=1}^{l} f\left(A_i\right),$$

$$f\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{j=1}^{k}(-1)^{j-1}\prod_{s=1}^{j} f\left(A_s\right).$$

by applying the inclusion-exclusion principle to a union of sets.

§1.2 GROUPS

1.2.1 Group. A set G with a binary operation $\circ$, denoted by $(G;\circ)$, is called a *group* if $x\circ y \in G$ for $\forall x, y \in G$ with conditions following hold:

(1) $(x\circ y)\circ z = x\circ(y\circ z)$ for $\forall x, y, z \in G$;

(2) There is an element 1_G, $1_G \in G$ such that $x\circ 1_G = x$;

(3) For $\forall x \in G$, there is an element $y, y \in G$, such that $x\circ y = 1_G$.

A group G is *Abelian* if the following additional condition holds.

(4) For $\forall x, y \in G$, $x\circ y = y\circ x$.

A set G with a binary operation $\circ$ satisfying the condition (1) is called to be a *semigroup*. Similarly, if it satisfies the conditions (1) and (4), then it is called an *Abelian semigroup*.

Example 1.2.1 The following sets with operations are groups:

(1) $(\mathbb{R};+)$ and $(\mathbb{R};\cdot)$, where R is the set of real numbers.

(2) $(U_2;\cdot)$, where $U_2 = \{1,-1\}$ and generally, $(U_n;\cdot)$, where $U_n = \left\{e^{i\frac{2\pi k}{n}}, k = 1, 2, \cdots, n\right\}$.

(3) For a finite set X, the set $SymX$ of all permutations on X with respect to permutation composition.

Clearly, the groups (1) and (2) are Abelian, but (3) is not in general.

Example 1.2.2 Let $GL(n,\mathbb{R})$ be the set of all invertible $n \times n$ matrixes with coefficients in $\mathbb{R}$ and $+, \cdot$ the ordinary matrix addition and multiplication. Then

(1) $(GL(n,\mathbb{R}); +)$ is an Abelian infinite group with identity $0_{n\times n}$, the $n\times n$ zero matrix and inverse $-A$ for $A \in GL(n,\mathbb{R})$, where $-A$ is the matrix replacing each entry a by $-a$ in matrix A.

(2) $(GL(n,\mathbb{R}); \cdot)$ is a non-Abelian infinite group if $n \geq 2$ with identity $1_{n\times n}$, the $n \times n$ unit matrix and inverse A^{-1} for $A \in GL(n,\mathbb{R})$, where $A \cdot A^{-1} = 1_{n\times n}$. For its non-Abelian, let $n = 2$ for simplicity and

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}.$$

Calculations show that

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 7 & -5 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 7 \end{bmatrix}.$$

Whence, $A \cdot B \neq B \cdot A$.

1.2.2 Group Property. A few properties of groups are listed in the following.

P1. *There is only one unit* $1_{\mathscr{G}}$ *in a group* $(\mathscr{G}; \circ)$.

In fact, if there are two units $1_{\mathscr{G}}$ and $1'_{\mathscr{G}}$ in $(\mathscr{G}; \circ)$, then we get $1_{\mathscr{G}} = 1_{\mathscr{G}} \circ 1'_{\mathscr{G}} = 1'_{\mathscr{G}}$, a contradiction.

P2. *There is only one inverse* a^{-1} *for* $a \in \mathscr{G}$ *in a group* $(\mathscr{G}; \circ)$.

If a_1^{-1}, a_2^{-1} both are the inverses of $a \in \mathscr{G}$, then we get that $a_1^{-1} = a_1^{-1} \circ a \circ a_2^{-1} = a_2^{-1}$, a contradiction.

P3. $(a^{-1})^{-1} = a,\ a \in \mathscr{G}$.

P4. *If* $a \circ b = a \circ c$ *or* $b \circ a = c \circ a$, *where* $a, b, c \in \mathscr{G}$, *then* $b = c$.

If $a \circ b = a \circ c$, then $a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$. According to the associative law, we get that $b = 1_{\mathscr{G}} \circ b = (a^{-1} \circ a) \circ b = a^{-1} \circ (a \circ c) = (a^{-1} \circ a) \circ c = 1_{\mathscr{G}} \circ c = c$. Similarly, if $b \circ a = c \circ a$, we can also get $b = c$.

P5. *There is a unique solution for equations* $a \circ x = b$ *and* $y \circ a = b$ *in a group* $(\mathscr{G}; \circ)$ *for* $a, b \in \mathscr{G}$.

Denote by $a^n = \underbrace{a \circ a \circ \cdots \circ a}_{n}$. Then the following property is obvious.

P6. *For any integers n, m and $a, b \in \mathscr{G}$, $a^n \circ a^m = a^{n+m}$, $(a^n)^m = a^{nm}$. Particularly, if $(\mathscr{G}; \circ)$ is Abelian, then $(a \circ b)^n = a^n \circ b^n$.*

1.2.3 Subgroup. A subset H of a group G is a *subgroup* if H is also a group under the same operation in G, denoted by $H \leq G$. The following results are well-known.

Theorem 1.2.1 *Let H be a subset of a group $(G; \circ)$. Then $(H; \circ)$ is a subgroup of $(G; \circ)$ if and only if $H \neq \emptyset$ and $a \circ b^{-1} \in H$ for $\forall a, b \in H$.*

Proof By definition if $(H; \circ)$ is a group itself, then $H \neq \emptyset$, there is $b^{-1} \in H$ and $a \circ b^{-1}$ is closed in H, i.e., $a \circ b^{-1} \in H$ for $\forall a, b \in H$.

Now if $H \neq \emptyset$ and $a \circ b^{-1} \in H$ for $\forall a, b \in H$, then,

(1) there exists an $h \in H$ and $1_G = h \circ h^{-1} \in H$;

(2) if $x, y \in H$, then $y^{-1} = 1_G \circ y^{-1} \in H$ and hence $x \circ (y^{-1})^{-1} = x \circ y \in H$;

(3) the associative law $x \circ (y \circ z) = (x \circ y) \circ z$ for $x, y, z \in H$ is hold in $(G; \circ)$. By (2), it is also hold in H. Thus, combining (1)-(3) enables us to know that $(H; \circ)$ is a group. □

Corollary 1.2.1 *Let $H_1 \leq G$ and $H_2 \leq G$. Then $H_1 \cap H_2 \leq G$.*

Proof Obviously, $1_G = 1_{H_1} = 1_{H_2} \in H_1 \cap H_2$. So $H_1 \cap H_2 \neq \emptyset$. Let $x, y \in H_1 \cap H_2$. Applying Theorem 1.2.2, we get that

$$x \circ y^{-1} \in H_1, \quad x \circ y^{-1} \in H_2.$$

Whence,

$$x \circ y^{-1} \in H_1 \cap H_2.$$

Thus, $(H_1 \cap H_2; \circ)$ is a subgroup of $(G; \circ)$. □

Theorem 1.2.2 (Lagrange) *Let $H \leq G$. Then $|G| = |H||G : H|$.*

Proof Let

$$G = \bigcup_{t \in G:H} t \circ H.$$

Notice that $t_1 \circ H \cap t_2 \circ H = \emptyset$ if $t_1 \neq t_2$ and $|t \circ H| = |H|$. We get that

$$|G| = \sum_{t \in G:H} t \circ H = |H||G : H|.$$ □

Let $H \leq G$ be a subgroup of G. For $\forall x \in G$, denote the sets $\{xh \mid \forall h \in H\}$, $\{hx \mid \forall h \in H\}$ by xH and Hx, respectively. A subgroup H of a group $(G\ ;\circ)$ is called a *normal subgroup* if for $\forall x \in G$, $xH = Hx$. Such a subgroup H is denoted by $H \lhd G$

For two subsets A, B of group $(G;\circ)$, the product $A \circ B$ is defined by

$$A \circ B = \{a \circ b |\ \forall a \in A,\ \forall b \in b\ \}.$$

Furthermore, if H is normal, i.e., $H \lhd G$, it can be verified easily that

$$(xH) \circ (yH) = (x \circ y)H \quad \text{and} \quad (Hx) \circ (Hy) = H(x \circ y)$$

for $\forall x, y \in G$. Thus the operation "$\circ$" is closed on the set $\{xH|x \in G\} = \{Hx|x \in G\}$. Such a set is usually denoted by G/H. Notice that

$$(xH \circ yH) \circ zH = xH \circ (yH \circ zH),\ \forall x, y, z \in G$$

and

$$(xH) \circ H = xH,\ (xH) \circ (x^{-1}H) = H.$$

Whence, G/H is also a group by definition, called a *quotient group*. Furthermore, we know the following result.

Theorem 1.2.3 *G/H is a group if and only if H is normal.*

Proof If H is a normal subgroup, then

$$(a \circ H)(b \circ H) = a \circ (H \circ b) \circ H = a \circ (b \circ H) \circ H = (a \circ b) \circ H$$

by the definition of normal subgroup. This equality enables us to check laws of a group following.

(1) Associative laws in G/H.

$$\begin{aligned} [(a \circ H)(b \circ H)](c \circ H) &= [(a \circ b) \circ c] \circ H = [a \circ (b \circ c)] \circ H \\ &= (a \circ H)[(b \circ H)(c \circ H)]. \end{aligned}$$

(2) Existence of identity element $1_{G/H}$ in G/H.

In fact, $1_{G/H} = 1 \circ H = H$.

(3) Inverse element for $\forall x \circ H \in G/H$.

Because of $(x^{-1} \circ H)(x \circ H) = (x^{-1} \circ x) \circ H = H = 1_{G/H}$, we know the inverse element of $x \circ H \in G/H$ is $x^{-1} \circ H$.

Conversely, if G/H is a group, then for $a \circ H, b \circ H \in G/H$, we have

$$(a \circ H)(b \circ H) = c \circ H.$$

Obviously, $a \circ b \in (a \circ H)(b \circ H)$. Therefore,

$$(a \circ H)(b \circ H) = (a \circ b) \circ H.$$

Multiply both sides by a^{-1}, we get that

$$H \circ b \circ H = b \circ H.$$

Notice that $1_G \in H$, we know that

$$b \circ H \subset H \circ b \circ H = b \circ H,$$

i.e., $b \circ H \circ b^{-1} \subset H$. Consequently, we also find $b^{-1} \circ H \circ b \subset H$ if replace b by b^{-1}, i.e., $H \subset b \circ H \circ b^{-1}$. Whence,

$$b^{-1} \circ H \circ b = H$$

for $\forall b \in G$. Namely, H is a normal subgroup of G. □

A *normal series* of a group $(G; \circ)$ is a sequence of normal subgroups

$$\{1_G\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_s = G,$$

where the $G_i,\ 1 \leq i \leq s$ are the *terms*, the quotient groups $G_{i+1}/G_i,\ 1 \leq i \leq s-1$ are the *factors* of the series and if all G_i are distinct, and the integer s is called the *length* of the series. Particularly, if the length $s = 2$, i.e., there are only normal subgroups $\{1_G\}$ and G in $(G; \circ)$, such a group $(G; \circ)$ is said to be *simple*.

1.2.4 Isomorphism Theorem. For two groups G, G', let σ be a mapping from G to G'. If

$$\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$$

for $\forall x, y \in G$, then σ is called a *homomorphism* from G to G'. Usually, a one to one homomorphism is called a *monomorphism* and an onto homomorphism an *epimorphism*. A homomorphism is a *bijection* if it is both one to one and onto. Two groups G, G' are said to be *isomorphic* if there exists a bijective homomorphism σ between them, denoted by $G \simeq G'$.

Some properties of homomorphism are listed following. They are easily verified by definition.

H1. *$\phi(x^n) = \phi^n(x)$ for all integers n, $x \in \mathscr{G}$, whence, $\phi(1_{\mathscr{G}}) = 1_{\mathscr{H}}$ and $\phi(x^{-1}) = \phi^{-1}(x)$.*

H2. *$o(\phi(x))|o(x)$, $x \in \mathscr{G}$.*

H3. *If $x \circ y = y \circ x$, then $\phi(x) \cdot \phi(y) = \phi(y) \cdot \phi(x)$.*

H4. $\mathrm{Im}\phi \leq \mathscr{H}$ and $\mathrm{Ker}\phi \lhd \mathscr{G}$.

Now let $\phi : G \rightarrow G'$ be a homomorphism. Its *image* $\mathrm{Im}\phi$ and *kernel* $\mathrm{Ker}\phi$ are respectively defined by

$$\mathrm{Im}\phi = G^{\phi} = \{\phi(x) \mid \forall x \in G\}$$

and

$$\mathrm{Ker}\phi = \{\, x \mid \forall x \in G,\ \phi(x) = 1_{G'}\,\}.$$

The following result, usually called the *homomorphism theorem* is well-known.

Theorem 1.2.4 *Let $\phi : G \rightarrow G'$ be a homomorphism of group. Then*

$$(G, \circ)/\mathrm{Ker}\phi \simeq \mathrm{Im}\phi.$$

Proof Notice that $\mathrm{Im}\phi \leq H$ and $\mathrm{Ker}\phi \lhd G$ by definition. So $G/\mathrm{Ker}\phi$ is a group by Theorem 1.2.3. We only need to check that ϕ is a bijection. In fact, $x \circ \mathrm{Ker}\phi \in \mathrm{Ker}\phi$ if and only if $x \in \mathrm{Ker}\phi$. Thus ϕ is an isomorphism from $G/\mathrm{Ker}\phi$ to $\mathrm{Im}\phi$. □

Corollary 1.2.1(Fundamental Homomorphism Theorem) *If $\phi : G \rightarrow H$ is an epimorphism, then $G/\mathrm{Ker}\phi$ is isomorphic to H.*

§1.3 RINGS

1.3.1 Ring. A set R with two binary operations "$+$" and "$\circ$", denoted by $(R; +, \circ)$, is said to be a *ring* if $x + y \in R$, $x \circ y \in R$ for $\forall x, y \in R$ such that the following conditions hold.

(1) $(R; +)$ is an Abelian group with unit 0, and in;

(2) $(R;\circ)$ is a semigroup;

(3) For $\forall x, y, z \in R$, $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$.

Denote the unit by 0, the inverse of a by $-a$ in the Abelian group $(R;+)$. A ring $(R;+,\circ)$ is *finite* if $|R| < +\infty$. Otherwise, *infinite*.

Example 1.3.1 Some examples of rings are as follows.

(1) $(\mathbb{Z};+,\cdot)$, where $\mathbb{Z}$ is the set of integers.

(2) $(p\mathbb{Z};+,\cdot)$, where p is a prime number and $p\mathbb{Z} = \{pn|n \in \mathbb{Z}\}$.

(3) $(\mathcal{M}_n(\mathbb{Z});+,\cdot)$, where $\mathcal{M}_n(\mathbb{Z})$ is the set of $n \times n$ matrices with each entry being an integer, $n \geq 2$.

Some elementary properties of rings $(R;+,\circ)$ can be found in the following:

R1. $0 \circ a = a \circ 0 = 0$ for $\forall a \in R$.

In fact, let $b \in R$ be an element in R. By $a \circ b = a \circ (b + 0) = a \circ b + a \circ 0$ and $b \circ a = (b + 0) \circ a = b \circ a + 0 \circ a$, we are easily know that $a \circ 0 = 0 \circ a = 0$.

R1. $(-a) \circ b = a \circ (-b) = -a \circ b$ and $(-a) \circ (-b) = a \circ b$ for $\forall a, b \in R$.

By definition, we are easily know that $(-a) \circ b + a \circ b = (-a + a) \circ b = 0 \circ b = 0$ in $(R;+,\circ)$. Thus $(-a) \circ b = -a \circ b$. Similarly, we can get that $a \circ (-b) = -a \circ b$. Consequently,

$$(-a) \circ (-b) = -a \circ (-b) = -(-a \circ b) = a \circ b.$$

R3. For any integer $n, m \geq 1$ and $a, b \in R$,

$$\begin{aligned}
&(n + m)a = na + ma,\\
&n(ma) = (nm)a,\\
&n(a + b) = na + nb,\\
&a^n \circ a^m = a^{n+m},\\
&(a^n)^m = a^{nm},
\end{aligned}$$

where $na = \underbrace{a + a + \cdots + a}_{n}$ and $a^n = \underbrace{a \circ a \circ \cdots \circ a}_{n}$.

All these identities can be verified by induction on the integer m. We only prove the last identity. For $m = 1$, we are easily know that $(a^n)^1 = (a^n) = a^{n1}$, i.e., $(a^n)^m = a^{nm}$ holds for $m = 1$. If it is held for $m = k \geq 1$, then

$$(a^n)^{k+1} = \left((a^n)^k\right) \circ (a^n)$$

$$\begin{aligned} &= a^{nk} \circ \left(\underbrace{a \circ a \circ \cdots \circ a}_{n} \right) \\ &= a^{nk+n} = a^{n(k+1)}. \end{aligned}$$

Thus $(a^n)^m = a^{nm}$ is held for $m = k + 1$. By the induction principle, we know it is true for any integer $n,\ m \geq 1$.

If R contains an element 1_R such that for $\forall x \in R$, $x \circ 1_R = 1_R \circ x = x$, we call R a *ring with unit*. All of these examples of rings in the above are rings with unit. For (1), the unit is 1, (2) is $\mathbb{Z}$ and (3) is $I_{n\times n}$.

The unit of $(R; +)$ in a ring $(R; +, \circ)$ is called *zero*, denoted by 0. For $\forall a, b \in R$, if

$$a \circ b = 0,$$

then a and b are called *divisors of zero*. In some rings, such as the $(\mathbb{Z}; +, \cdot)$ and $(p\mathbb{Z}\ ; +, \cdot)$, there must be a or b be 0. We call it only has a *trivial divisor of zero*. But in the ring $(pq\mathbb{Z}; +, \cdot)$ with p, q both being prime, since

$$p\mathbb{Z} \cdot q\mathbb{Z} = 0$$

and $p\mathbb{Z} \neq 0$, $q\mathbb{Z} \neq 0$, we get non-zero divisors of zero, which is called to have *non-trivial divisors of zero*. The ring $(\mathcal{M}_n(\mathbb{Z}); +, \cdot)$ also has non-trivial divisors of zero

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = O_{n\times n}.$$

A *division ring* is a ring which has no non-trivial divisors of zero. The integer ring $(\mathbb{Z}; +, \cdot)$ is a divisor ring, but the matrix ring $(\mathcal{M}_n(\mathbb{Z}); +, \cdot)$ is not. Furthermore, if $(R \setminus \{0\}; \circ)$ is a group, then the ring $(R; +, \circ)$ is called a *skew field*. Clearly, a skew field is a divisor ring by properties of groups.

1.3.2 Subring. A non-empty subset R' of a ring $(R; +, \circ)$ is called a *subring* if $(R'; +, \circ)$ is also a ring. The following result for subrings can be obtained immediately by definition.

Theorem 1.3.1 *Let $R' \subset R$ be a subset of a ring $(R; +, \circ)$. If $(R'; +)$ is a subgroup of $(R; +)$ and R' is closed under the operation "$\circ$", then $(R'; +, \circ)$ is a subring of $(R, +.\circ)$.*

Proof Because $R' \subset R$ and $(R; +, \circ)$ is a ring, we know that $(R'; \circ)$ is also a semigroup and the distribute lows $x \circ (y+z) = x \circ y + x \circ z$, $(x+y) \circ z = x \circ z + y \circ z$ hold for $\forall x, y, z \in R'$. Thus $()R'; +, \circ$ is also a ring. $\square$

Combining Theorems 1.3.1 with 1.2.1, we know the following criterion for subrings of a ring.

Theorem 1.3.2 *Let* $R' \subset R$ *be a subset of a ring* $(R; +, \circ)$. *If* $a - b,\ a \cdot b \in R'$ *for* $\forall a, b \in R'$, *then* $(R'; +, \circ)$ *is a subring of* $(R, +. \circ)$.

Example 1.3.2 Let $(M_3(\mathbb{Z}); +, \cdot)$ be the ring determined in Example 1.3.1(3). Then all matrixes with following forms

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}$$

consist of a subring of $(M_3(\mathbb{Z}); +, \cdot)$.

1.3.3 Commutative Ring. A *commutative ring* is such a ring $(R; +, \circ)$ that $a \circ b = b \circ a$ for $\forall a, b \in R$. Furthermore, if $(R \setminus \{0\}; \circ)$ is an Abelian group, then $(R; +, \circ)$ is called a *field*. For example, the rational number ring $(\mathbb{N}; +, \cdot)$ is a field.

A commutative ring $(R; +, \circ)$ is called an *integral domain* is there are no non-trivial divisors of zero in R. We know the following result for finite integral domains.

Theorem 1.3.3 *Any finite integral domain is a field.*

Proof Let $(R; +\circ)$ be a finite integral domain with $R = \{a_1 = 1_R, a_2, \cdots, a_n\}$, $b \in R$ and a sequence

$$b \circ a_1, b \circ a_2, \cdots, b \circ a_n.$$

Then for any integer $i \neq j,\ 1 \leq i, j \leq m$, $b \circ a_i \neq b \circ a_j$. Otherwise, we get $b \circ (a_i - a_j) = 0$ with $a \neq 0$ and $a_i - a_j \neq 0$. Contradicts to the definition of integral domain. Therefore,

$$R = \{b \circ a_1, b \circ a_2, \cdots, b \circ a_n\}.$$

Consequently, there must be an integer $k,\ 1 \leq k \leq n$ such that $b \circ a_k = 1_R$. Thus $b^{-1} = a_k$. This implies that $(R \setminus \{0\}; \circ)$ is a group, i.e., $(R; +\circ)$ is a field. $\square$

Let D be an integral domain. Define the quotient field $Q[D]$ by

$$Q[D] = \{\ (a, b) \mid a, b \in D,\ b \neq 0\ \}$$

with the convention that

$$(a,b) \equiv (a',b') \qquad \text{if and only if} \qquad ab' = a'b.$$

Define operations of sums and products respectively by

$$(a,b) + (a',b') = (ab' + a'b, bb')$$
$$(a,b) \cdot (a',b') = (aa', bb').$$

Theorem 1.3.4 *$Q[D]$ is a field for any integral domain D.*

Proof It is easily to verify that $Q[D]$ is also an integral domain with identity elements $(0,1)$ for addition and $(1,1)$ for multiplication. We prove that there exists an inverse for every element $u \neq 0$ in $Q[D]$. In fact, for $(a,b) \not\equiv (0,1)$,

$$(a,b) \cdot (b,a) = (ab, ab) \equiv (1,1).$$

Thus $(a,b)^{-1} = (b,a)$. Whence, $Q[D]$ is a field by definition. □

For seeing D is actually a subdomain of $Q[D]$, associate each element $a \in D$ with $(a,1) \in Q[D]$. Then it is easily to verify that

$$(a,1) + (b,1) = (a \cdot 1 + b \cdot 1, 1 \cdot 1) = (a+b, 1),$$
$$(a,1) \cdot (b,1) = (ab, 1 \cdot 1) = (ab, 1),$$
$$(a,1) \equiv (b,1) \quad \text{if and only if} \quad a = b.$$

Thus the 1-1 mapping $a \leftrightarrow (a,1)$ is an isomorphism between the domain D and a subdomain $\{ (a,1) \mid a \in D \}$ of $Q[D]$. We get a result following.

Theorem 1.3.5 *Any integral domain D can be embedded isomorphically in a field $Q[D]$. Particularly, let $D = \mathbb{Z}$. Then the integral domain $\mathbb{Z}$ can be embedded in $Q[\mathbb{Z}] = \mathbb{Q}$.*

1.3.4 Ideal. An *ideal* I of a ring $(R; +, \circ)$ is a non-void subset of R with properties:

(1) $(I; +)$ is a subgroup of $(R; +)$;

(2) $a \circ x \in I$ and $x \circ a \in I$ for $\forall a \in I, \forall x \in R$.

Let $(R; +, \circ)$ be a ring. A chain

$$R \succ R_1 \succ \cdots \succ R_l = \{1_\circ\}$$

satisfying that R_{i+1} is an ideal of R_i for any integer $i, 1 \leq i \leq l$, is called an *ideal chain* of $(R, +, \circ)$. A ring whose every ideal chain only has finite terms is called an *Artin ring*. Similar to the case of normal subgroup, consider the set $x + I$ in the group $(R; +)$. Calculation shows that $R/I = \{x + I | \ x \in R\}$ is also a ring under these operations "+" and "∘", called a *quotient ring* of R to I.

For two rings rings $(R; +, \circ), (R'; *, \bullet)$, let $\iota : \ R \to R'$ be a mapping from R to R'. If

$$\iota(x + y) = \iota(x) * \iota(y),$$
$$\iota(x \circ y) = \iota(x) \bullet \iota(y)$$

for $\forall x, y \in R$, then ι is called a *homomorphism* from $(R; +, \circ)$ to $(R'; *, \bullet)$. Furthermore, if ι is an objection, then ring $(R; +, \circ)$ is said to be *isomorphic* to ring $(R'; *, \bullet)$ and denoted by rings $(R; +, \circ) \simeq (R'; *, \bullet)$. Similar to Theorem 1.2.4, we know the following result.

Theorem 1.3.6 *Let* $\iota : \ R \to R'$ *be a homomorphism from* $(R; +, \circ)$ *to* $(R'; *, \bullet)$*. Then*

$$(R; +, \circ)/Ker\iota \simeq Im\iota.$$

§1.4 VECTOR SPACES

1.4.1 Vector Space. A *vector space* or *linear space* consists of the following:

(1) A field F of scalars;

(2) A set V of objects, called vectors;

(3) An operation, called vector addition, which associates with each pair of vectors $\mathbf{a}, \mathbf{b}$ in V a vector $\mathbf{a} + \mathbf{b}$ in V, called the sum of $\mathbf{a}$ and $\mathbf{b}$, in such a way that

(a) Addition is commutative, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$;

(b) Addition is associative, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$;

(c) There is a unique vector $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in V;

(d) For each vector $\mathbf{a}$ in Vthere is a unique vector $-\mathbf{a}$ in V such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$;

(4) An operation "·", called scalar multiplication, which associates with each scalar k in F and a vector $\mathbf{a}$ in V a vector $k \cdot \mathbf{a}$ in V, called the product of k with $\mathbf{a}$, in such a way that

(a) $1 \cdot \mathbf{a} = \mathbf{a}$ for every $\mathbf{a}$ in V;
(b) $(k_1k_2) \cdot \mathbf{a} = k_1(k_2 \cdot \mathbf{a})$;
(c) $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$;
(d) $(k_1 + k_2) \cdot \mathbf{a} = k_1 \cdot \mathbf{a} + k_2 \cdot \mathbf{a}$.

We say that V is a *vector space over the field F, denoted by* $(V\ ; +, \cdot)$.

Example 1.4.1 Two vector spaces are listed in the following.

(1) **The n-tuple space R^n over the real number field R.** Let V be the set of all n-tuples $(x_1, x_2, \cdots, x_n)$ with $x_i \in R, 1 \leq i \leq n$. If $\forall \mathbf{a} = (x_1, x_2, \cdots, x_n)$, $\mathbf{b} = (y_1, y_2, \cdots, y_n) \in V$, then the sum of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} + \mathbf{b} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

The product of a real number k with $\mathbf{a}$ is defined by

$$k\mathbf{a} = (kx_1, kx_2, \cdots, kx_n).$$

(2) **The space $Q^{m\times n}$ of $m\times n$ matrices over the rational number field Q.** Let $Q^{m\times n}$ be the set of all $m \times n$ matrices over the natural number field Q. The sum of two vectors A and B in $Q^{m\times n}$ is defined by

$$(A + B)_{ij} = A_{ij} + B_{ij},$$

and the product of a rational number p with a matrix A is defined by

$$(pA)_{ij} = pA_{ij}.$$

1.4.2 Vector Subspace. Let V be a vector space over a field F. A *subspace* W of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V. The following result for subspaces of a vector space is easily obtained.

Theorem 1.4.1 *A non-empty subset W of a vector space $(V; +, \cdot)$ over the field F is a subspace of $(V; +, \cdot)$ if and only if for each pair of vectors* $\mathbf{a}$, $\mathbf{b}$ *in W and each scalar α in F the vector* $\alpha \cdot \mathbf{a} + \mathbf{b}$ *is also in W.*

Proof Let W be a non-empty subset of V such that $\alpha \cdot \mathbf{a} + \mathbf{b}$ belongs to W for $\forall \mathbf{a}$, $\mathbf{b} \in V$ and all scalars α in F. Notice that $W \neq \emptyset$, there are a vector $\mathbf{x} \in W$. By assumption, we get that $(-1)\mathbf{x} + \mathbf{x} = \mathbf{0} \in W$. Hence, $\alpha\mathbf{x} + \mathbf{0} = \alpha\mathbf{x} \in W$ for $\mathbf{x} \in W$ and $\alpha \in F$. Particularly, $(-1)\mathbf{x} = -\mathbf{x} \in W$. Finally, if $\mathbf{x},\ \mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$. Thus W is a subspace of V.

Conversely, if $W \subset V$ is a subspace of V, $\mathbf{a},\ \mathbf{b}$ in W and α is scalar F, then $\alpha\cdot\mathbf{a}+\mathbf{b} \in W$ by definition. □

This theorem enables one to get the following result.

Theorem 1.4.2 *Let V be a vector space over a field F. Then the intersection of any collection of subspaces of V is a subspace of V.*

Proof Let $W = \bigcap\limits_{i\in I} W_i$, where W_i is a subspace of V for each $i \in I$. First, we know that $\mathbf{0} \in W_i$ for $i \in I$ by definition. Whence, $\mathbf{0} \in W$. Now let $\mathbf{a},\ \mathbf{b} \in W$ and $\alpha \in F$. Then $\mathbf{a},\ \mathbf{b} \in W_i$ for $W \subset W_i$ for $\forall i \in I$. According Theorem 1.4.1, we know that $\alpha \cdot \mathbf{a} + \mathbf{b} \in W_i$. So $\alpha \cdot \mathbf{a} + \mathbf{b} \in \bigcap\limits_{i\in I} W_i = W$. Whence, W is a subspace of V by Theorem 1.4.1. □

Let U be a set of some vectors in a vector space V over F. The subspace spanned by U is defined by

$$\langle U\rangle = \{\, \alpha_1 \cdot \mathbf{a_1} + \alpha_2 \cdot \mathbf{a_2} + \cdots + \alpha_l \cdot \mathbf{a}_l \mid l \geq 1, \alpha_i \in F, \text{ and } \mathbf{a_j} \in S, 1 \leq i \leq l \,\}.$$

A subset S of V is said to be *linearly dependent* if there exist distinct vectors $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}$ in S and scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ in F, not all of which are 0, such that

$$\alpha_1 \cdot \mathbf{a_1} + \alpha_2 \cdot \mathbf{a_2} + \cdots + \alpha_n \cdot \mathbf{a_n} = \mathbf{0}.$$

A set which is not linearly dependent is usually called *linearly independent*, i.e., for distinct vectors $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}$ in S if there are scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ in F such that

$$\alpha_1 \cdot \mathbf{a_1} + \alpha_2 \cdot \mathbf{a_2} + \cdots + \alpha_n \cdot \mathbf{a_n} = \mathbf{0},$$

then $\alpha_i = 0$ for integers $1 \leq i \leq n$.

Let V be a vector space over a field F. A *basis* for V is a linearly independent set of vectors in V which spans the space V. Such a space V is called *finite-dimensional* if it has a finite basis.

Theorem 1.4.3 *Let V be a vector space spanned by a finite set of vectors $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_m}$. then each independent set of vectors in V is finite, and contains no more than m elements.*

Proof Let S be a set of V containing more than m vectors. We only need to show that S is linearly dependent. Choose $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n} \in S$ with $n > m$. Since $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_m}$

span V, there must exist scalars $A_{ij} \in F$ such that

$$\mathbf{x}_j = \sum_{i=1}^{m} A_{ij}\mathbf{a}_i.$$

Whence, for any n scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$, we get that

$$\begin{aligned} \alpha_1\mathbf{x}_1 + \alpha\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n &= \sum_{j=1}^{n}\alpha_j\sum_{i=1}^{m}A_{ij}\mathbf{a}_i \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(A_{ij}\alpha_j\right)\mathbf{a}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n}A_{ij}\alpha_j\right)\mathbf{a}_i. \end{aligned}$$

Notice that $n > m$. There exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ not all 0 such that

$$\sum_{j=1}^{n} A_{ij}\alpha_j = 0, \qquad 1 \leq i \leq m.$$

Thus $\alpha_1\mathbf{x}_1 + \alpha\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}$. Whence, S is linearly dependent. □

Theorem 1.4.3 enables one knowing the following consequences.

Corollary 1.4.1 *If V is a finite-dimensional, then any two bases of V have the same number of vectors.*

Proof Let $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_m}$ be a basis of V. according to Theorem 1.4.3, every basis of V is finite and contains no more than m vectors. Thus if $\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}$ is a basis of V, then $n \leq m$. Similarly, e also know that $m \leq n$. Whence, $n = m$. □

This consequence allows one to define the dimension $\dim V$ of a finite-dimensional vector space as the number of elements in a basis of V.

Corollary 1.4.2 *Let V be a finite-dimensional vector space with $n = \dim V$. Then no subset of V containing fewer than n vectors can span V.*

Let $\dim V = n < +\infty$. An *ordered basis* for V is a finite sequence $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ of vectors which is linearly independent and spans V. Whence, for any vector $\mathbf{x} \in V$, there is an n-tuple $(x_1, x_2, \cdots, x_n)$ such that

$$\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{a}_i.$$

The n-tuple is unique, because if there is another n-tuple $(z_1, z_2, \cdots, z_n)$ such that

$$\mathbf{x} = \sum_{i=1}^{n} z_i\mathbf{a}_i,$$

Then there must be

$$\sum_{i=1}^{n}(x_i - z_i)\,\mathbf{a}_i = \mathbf{0}.$$

We get $z_i = x_i$ for $1 \leq i \leq n$ by the linear independence of $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$. Thus each ordered basis for V determines a 1-1 correspondence

$$\mathbf{x} \leftrightarrow (x_1, x_2, \cdots, x_n)$$

between the set of all vectors in V and the set of all n-tuples in $F^n = \underbrace{F \times F \times \cdots \times F}_{n}$.

The following result shows that the dimensions of subspaces of a finite-dimensional vector space is finite.

Theorem 1.4.4 *If W is a subspace of a finite-dimensional vector space V, then every linearly independent subset of W is finite and is part of basis for W.*

Proof Let $S_0 = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ be a linearly independent subset of W. By Theorem 1.4.3, $n \leq \dim W$. We extend S_0 to a basis for W. If S_0 spans W, then S_0 is a basis of W. Otherwise, we can find a vector $\mathbf{b}_1 \in W$ which can not be spanned by elements in S_0. Then $S_0 \cup \{\mathbf{b}_1\}$ is also linearly independent. Otherwise, there exist scalars $\alpha_0,\ \alpha_i,\ 1 \leq i \leq |S_0|$ with $\alpha_0 \neq 0$ such that

$$\alpha_0\mathbf{b}_1 + \sum_{i=1}^{|S_0|}\alpha_i\mathbf{a}_i = 0.$$

Whence

$$\mathbf{b}_1 = -\frac{1}{\alpha_0}\sum_{i=1}^{|S_0|}\alpha_i\mathbf{a}_i,$$

a contradiction.

Let $S_1 = S_0 \cup \{\mathbf{b}_1\}$. If S_1 spans W, we get a basis of W containing S_0. Otherwise, we can similarly find a vector $\mathbf{b}_2$ such that $S_2 = S_0 \cup \{\mathbf{b}_1, \mathbf{b}_2\}$ is linearly independent. Continue i this way, we can get a set

$$S_m = S_0 \cup \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m\}$$

in at more than $\dim W - n \leq \dim V - n$ step such that S_m is a basis for W. □

For two subspaces U, W of a space V, the sum of subspaces U, W is defined by

$$U + W = \{\, \mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W \,\}.$$

Then, we have results in the following result.

Theorem 1.4.5 *If* W_1 *and* W_2 *are finite-dimensional subspaces of a vector space* V*, then* $W_1 + W_2$ *is finite-dimensional and*

$$\dim W_1 + \dim W_2 = \dim\left(W_1 \bigcap W_2\right) + \dim\left(W_1 + W_2\right).$$

Proof According to Theorem 1.4.4, $W_1 \cap W_2$ has a finite basis $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k\}$ which is part of a basis $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k, \mathbf{b}_1, \cdots, \mathbf{b}_l\}$ for W_1 and part of a basis $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k, \mathbf{c}_1, \cdots, \mathbf{c}_m\}$ for W_2. Clearly, $W_1 + W_2$ is spanned by vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k,\ \mathbf{b}_1, \cdots, \mathbf{b}_l,\ \mathbf{c}_1, \cdots, \mathbf{c}_m$. If there are scalars α_i, β_j and γ_r, $1 \le i \le k$, $1 \le j \le l$, $1 \le r \le m$ such that

$$\sum_{i=1}^{k} \alpha_i \mathbf{a}_i + \sum_{j=1}^{l} \beta_j \mathbf{b}_j + \sum_{r=1}^{m} \gamma_r \mathbf{c}_r = \mathbf{0},$$

then

$$-\sum_{r=1}^{m} \gamma_r \mathbf{c}_r = \sum_{i=1}^{k} \alpha_i \mathbf{a}_i + \sum_{j=1}^{l} \beta_j \mathbf{b}_j,$$

which implies that $\mathbf{v} = \sum\limits_{r=1}^{m} \gamma_r \mathbf{c}_r$ belongs to W_1. Because $\mathbf{v}$ also belongs to W_2 it follows that $\mathbf{v}$ belongs to $W_1 \cap W_2$. So there are scalars $\delta_1, \delta_2, \cdots, \delta_k$ such that

$$\mathbf{v} = \sum_{r=1}^{m} \gamma_r \mathbf{c}_r = \sum_{i=1}^{k} \delta_i \mathbf{a}_i.$$

Notice that $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k, \mathbf{c}_1, \cdots, \mathbf{c}_m\}$ is linearly independent. There must be $\gamma_r = 0$ for $1 \le r \le m$. We therefore get that

$$\sum_{i=1}^{k} \alpha_i \mathbf{a}_i + \sum_{j=1}^{l} \beta_j \mathbf{b}_j = \mathbf{0}.$$

But $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k, \mathbf{b}_1, \cdots, \mathbf{b}_l\}$ is also linearly independent. We get also that $\alpha_i = 0,\ 1 \le i \le k$ and $\beta_j = 0,\ 1 \le j \le l$. Thus

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k,\ \mathbf{b}_1, \cdots, \mathbf{b}_l,\ \mathbf{c}_1, \cdots, \mathbf{c}_m\}$$

is a basis for $W_1 + W_2$. Counting numbers in this basis for $W_1 + W_2$, we get that

$$\begin{aligned} \dim W_1 + \dim W_2 &= (k + l) + k + m = k + (k + l + m) \\ &= \dim\left(W_1 \bigcap W_2\right) + \dim\left(W_1 + W_2\right). \end{aligned}$$

This completes the proof. □

1.4.3 Linear Transformation. Let V and W be vector spaces over a field F. A *linear transformation* from V to W is a mapping T from V to W such that

$$T(\alpha\mathbf{a}+\mathbf{b})=\alpha(T(\mathbf{a}))+T(\mathbf{b})$$

for all $\mathbf{a},\mathbf{b}$ in V and all scalars α in F. If such a linear transformation is 1-1, the space V is called *linear isomorphic* to W, denoted by $V\overset{l}{\simeq}W$.

Theorem 1.4.6 *Every finite-dimensional vector space V over a field F is isomorphic to space* F^n, *i.e.,* $V\overset{l}{\simeq}F^n$, *where* $n=dimV$.

Proof Let $\{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n\}$ be an ordered basis for V. Then for any vector $\mathbf{x}$ in V, there exist scalars $x_1,x_2,\cdots,x_n$ such that

$$\mathbf{x}=x_1\mathbf{a}_1+x_2\mathbf{a}_2+\cdots+x_n\mathbf{a}_n.$$

Define a linear mapping from V to F^n by

$$T:\ \mathbf{x}\leftrightarrow(x_1,x-2,\cdots,x_n).$$

Then such a mapping T is linear, 1-1 and mappings V onto F^n. Thus $V\overset{l}{\simeq}F^n$. □

Let $\{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n\}$ and $\{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_m\}$ be ordered bases for vectors V and W, respectively. Then a linear transformation T is determined by its action on $\mathbf{a}_j$, $1\leq j\leq n$. In fact, each $T(\mathbf{a}_j)$ is a linear combination

$$T(\mathbf{a}_j)=\sum_{i=1}^{m}A_{ij}\mathbf{b}_i$$

of $\mathbf{b}_i$, the scalars $A_{1j},A_{2j},\cdots,A_{mj}$ being the coordinates of $T(\mathbf{a}_j)$ in the ordered basis $\{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_m\}$. Define an $m\times n$ matrix by $A=[A_{ij}]$ with entry A_{ij} in the position (i,j). Such a matrix is called a *transformation matrix*, denoted by $A=[T]_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}$.

Now let $\mathbf{a}=\alpha_1\mathbf{a}_1+\alpha_2\mathbf{a}_2+\cdots+\alpha_n\mathbf{a}_n$ be a vector in V. Then

$$\begin{aligned}T(\mathbf{a})&=T\left(\sum_{j=1}^{n}\alpha_j\mathbf{a}_j\right)=\sum_{j=1}^{n}\alpha_jT\left(\mathbf{a}_j\right)\\&=\sum_{j=1}^{n}\alpha_j\sum_{i=1}^{m}A_{ij}\mathbf{b}_i=\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\alpha_jA_{ij}\alpha_j\right)\mathbf{b}_i.\end{aligned}$$

Whence, if X is the coordinate matrix of $\mathbf{a}$ in the ordered basis $\{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_m\}$, then AX is the coordinate matrix of $T(\mathbf{a})$ in the same basis because the scalars $\sum_{j=1}^{n}A_{ij}\alpha_j$ is the entry in the ith row of the column matrix AX. On the other hand, if A is an $m \times n$ matrix over a field F, then

$$T\left(\sum_{j=1}^{n}\alpha_j\mathbf{a}_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\alpha_j A_{ij}\alpha_j\right)\mathbf{b}_i$$

indeed defines a linear transformation T from V into W with a transformation matrix A. This enables one getting the following result.

Theorem 1.4.7 *Let* $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ *and* $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m\}$ *be ordered bases for vectors V and W over a field F, respectively. Then for each linear transformation T from V into W, there is an $m \times n$ matrix A with entries in F such that*

$$[T(\mathbf{a})]_{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_m} = A\,[\mathbf{a}]_{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_m}$$

for every $\mathbf{a}$ *in V. Furthermore, $T \to A$ is a 1-1 correspondence between the set of all linear transformations from V into W and the set of all $m \times n$ matrix over F.*

Let V be a vector space over a field F. A *linear operator* of V is a linear transformation from V to V. Calculation can show easily the following result.

Theorem 1.4.8 *Let V be a vector space over a field F with ordered bases* $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ *and* $\{\mathbf{a}'_1, \mathbf{a}'_2, \cdots, \mathbf{a}'_n\}$ *and T a linear operator on V. If $A = [A_1, A_2, \cdots, A_n]$ is the $n \times n$ matrix with columns $A_j = [\mathbf{a}'_j]_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}$, then*

$$[T]_{\mathbf{a}'_1,\mathbf{a}'_2,\cdots,\mathbf{a}'_n} = A^{-1}\,[T]_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}\,A.$$

Generally, if T' is an invertible operator on V determined by $T'(\mathbf{a}_j) = \mathbf{a}'_j$ for $j = 1, 2, \cdots, n$, then

$$[T]_{\mathbf{a}'_1,\mathbf{a}'_2,\cdots,\mathbf{a}'_n} = [T']^{-1}_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}[T]_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}[T']_{\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n}.$$

§1.5 METRIC SPACES

1.5.1 Metric Space. A *metric space* $(X; d)$ is a set X associated with a metric function $d : M \times M \to R^+ = \{x \mid x \in R, x \geq 0\}$ with conditions following hold for $\forall x, y, z \in M$.

(1)(*definiteness*) $d(x, y) = 0$ if and only if $x = y$;

(2)(*symmetry*) $d(x, y) = d(y, x)$;

(3)(*triangle inequality*) $d(x,y) + d(y,z) \geq d(x,z)$.

Example 1.5.1 Euclidean Space $\mathbf{R}^n$.

Let $\mathbf{R}^n = \{\ (x_1, x_2, \cdots, x_n) \mid x_i \in \mathbf{R}, 1 \leq i \leq n\ \}$. For $\forall \mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbf{R}^n$, define

$$\rho(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)}.$$

Then d is a metric on $\mathbf{R}^n$.

Clearly, conditions (1) and (2) are true. We only need to verify the condition (3). Notice that

$$\sum_{i=1}^{n} b_i^2 + 2x \sum_{i=1}^{n} a_i b_i + x^2 \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n} (a_i + x b_i)^2 \geq 0.$$

Consequently, the discriminant

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 - \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) \leq 0.$$

Thus

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

Applying this inequality, we know that

$$\begin{aligned}
\sum_{i=1}^{n} (a_i + b_i)^2 &= \sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \\
&\leq \sum_{i=1}^{n} a_i^2 + 2 \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} a_i^2} + \sum_{i=1}^{n} b_i^2 \\
&= \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2} \right)^2 .
\end{aligned}$$

Let $a_i = x_i - y_i$, $b_i = y_i - z_i$. Then $x_i - z_i = a_i + b_i$ for integers $1 \leq i \leq n$. Substitute these numbers in the previous inequality, we get that

$$\sum_{i=1}^{n} (x_i - z_i)^2 \leq \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \right)^2 .$$

Thus $d(x,z) \leq d(x,y) + d(y,z)$.

Example 1.5.2 If $(X; d)$ is a metric space. Define

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Then $(X; d_1)$ is also a metric space. In fact, by noting that the function $g(x) = \dfrac{x}{1+x}$ is an increasing function for $x \geq 0$, it is easily to verify that conditions (1) − (3) hold.

1.5.2 Convergent Sequence. Any $x, x \in X$ is called a point of $(X; d)$. A sequence $\{x_n\}$ is said to be *convergent to* x if for any number $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $d(x_n, x) < \epsilon$, denoted by $\lim\limits_{n} x_n = x$. We have known the following results.

Theorem 1.5.1 *Any sequence* $\{x_n\}$ *in a metric space has at most one limit point.*

Proof Otherwise, if $\{x_n\}$ has two limit points $\lim\limits_{n\to\infty} x_n \to x$ and $\lim\limits_{n\to\infty} x_n \to x'$, then

$$0 \leq d(x, x') \leq d(x_n, x) + d(x_n, x')$$

for an integer $n \geq 1$. Let $n \to \infty$. then $d(x, x') = 0$. Thus $x = x'$ by the condition (1). □

Theorem 1.5.2 *Let* $(X; d)$ *be a metric space. If* $x_n \to x_0$ *and* $y_n \to y_0$*, then* $d(x_n, y_n) \to d(x_0, y_0)$ *when* $n \to \infty$*, i.e.,* $d(x, y)$ *is continuous.*

Proof Applying the condition (3), we get inequalities

$$d(x_n, y_n) \leq d(x_n, x_0) + d(x_0, y_0) + d(y_n, y_0)$$

and

$$d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_n) + d(y_n, y_0).$$

Whence,

$$|d(x_n, y_n) - d(x_0, y_0)| \leq d(x_n, x_0) + d(y_n, y_0) \to 0$$

if $n \to \infty$. Thus $d(x_n, y_n) \to d(x_0, y_0)$ when $n \to \infty$. □

For $x_0 \in X$ and $\epsilon > 0$, an ϵ-disk about x_0 is defined by

$$B(x_0, \epsilon) = \{\, x \mid x \in X, d(x, x_0) < \epsilon\}.$$

If $A \subset X$ and there is an ϵ-disk $B(x_0, \epsilon) \supset A$, we say A is a bounded point set of X.

Theorem 1.5.3 *Any convergent sequence* $\{x_n\}$ *in a metric space* $(X; d)$ *is bounded.*

Proof Let $x_n \to x_0$ when $n \to \infty$ and $\varepsilon = 1$. Then there exists an integer N such that for any integer $n > N$, $d(x_n, x_0) < 1$. Denote $c = \max\{d(x_1, x_0), d(x_2, x_0), \cdots, d(x_N, x_0)\}$. We get that

$$d(x_n, x_0) < 1 + c, \quad n = 1, 2, \cdots k, \cdots.$$

Let $R = 1 + c$. Then $\{x_n\} \subset B(x_0, R)$. □

1.5.3 Completed Space. Let $(X; d)$ be a metric space and $\{x_n\}$ a sequence in X. If for any number $\varepsilon > 0, \varepsilon \in \mathbf{R}$, there is an integer N such that $n, m \geq N$ implies $\rho(x_n, x_m) < \varepsilon$, then we call $\{x_n\}$ a *Cauchy sequence*. A metric space $(X; d)$ is *completed* if its every Cauchy sequence converges.

Theorem 1.5.3 *For a completed metric space $(X; d)$, if an ε-disk sequence $\{B_n\}$ satisfies*

(1) $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$;
(2) $\lim\limits_{n} \varepsilon_n = 0$,

where $\varepsilon_n > 0$ and $B_n = \{\, x \mid x \in X, d(x, x_n) \leq \varepsilon_n\}$ for any integer $n, n = 1, 2, \cdots$, then $\bigcap\limits_{n=1}^{\infty} B_n$ only has one point.

Proof First, we prove the sequence $\{x_n\}$ consisting of centers of ε-disk B_n is an Cauchy sequence. In fact, by the condition (1), if $m \geq n$, then $x_m \in B_m \subset B_n$. Thus $d(x_m, x_n) \leq \varepsilon_n$. According to the condition (2), for any positive number $\varepsilon > 0$, there exists an integer N such that $\varepsilon_n < \varepsilon$ if $n > N$. Whence, if $m, n > N$, there must be that $d(x_m, x_n) < \varepsilon$, i.e., $\{x_n\}$ is a Cauchy sequence.

By assumption, $(X; d)$ is completed. We know that $\{x_n\}$ convergent to a point $x_0 \in X$. Let $m \to \infty$ in the inequality $d(x_m, x_n) \leq \varepsilon_n$. We get that $d(x_, x_n) \leq \varepsilon_n$ for all $n = 1, 2, \cdots$. Whence, $x_0 \in B_n$ for all integers $n \geq 1$. Thus $x_0 \in \bigcap\limits_{i=1}^{\infty} B_n$.

If there exists another point $y \in \bigcap\limits_{i=1}^{\infty} B_n$, there must be $d(y, x_n) \leq \varepsilon_n$ for $n = 1, 2, \cdots$. By Theorem 1.5.2, we have that

$$0 \leq d(y, x_0) = \lim_{n} \leq \lim_{n} \varepsilon_n = 0.$$

Thus $d(y, x_0) = 0$, i.e., $y = x_0$. □

For a metric space $(X; d)$ and a mapping $T : X \to X$ on $(X; d)$, if there exists a point $x^* \in X$ such that

$$Tx^* = x^*,$$

then x^* is called a *fixed point* of T. If there exists a constant $\eta,\ 0 < \eta < 1$ such that

$$\rho(Tx, Ty) \leq \eta d(x, y)$$

for $\forall x,\ y \in X$, then T is called a *contraction*.

Theorem 1.5.4 (Banach) *Let $(X; d)$ be a completed metric space and let $T : X \to X$ be a contraction. Then T only has one fixed point.*

Proof Choose $x_0 \in X$. Let

$$x_1 = T(x_0),\ x_2 = T(x_1), \cdots, x_{n+1} = T(x_n), \cdots.$$

We prove first such a sequence $\{x_n\}$ is a Cauchy sequence. In fact, for integers $m,\ n$, $m < n$, by

$$\begin{aligned} d(x_{m+1}, x_m) &= d(T(x_m), d(x_{m-1})) \leq \eta d(x_m, x_{m-1}) \\ &\leq \eta^2 d(x_{m-1}, x_{m-2}) \leq \cdots \leq \eta^m d(x_1, x_0). \end{aligned}$$

we know that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m=1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \left(\eta^m + \eta^{m-1} + \cdots + \eta^{n-1}\right) d(x_1, x_0) \\ &= \eta^m \times \frac{1 - \eta^{n-m}}{1 - \eta} d(x_1, x_0) \\ &\leq \frac{\eta^m d(x_1, x_0)}{1 - \eta} \to 0 \quad (\text{if } m,\ n \to 0). \end{aligned}$$

Because $(X; d)$ is completed, there must exists a point $x^* \in X$ such that $x_n \to x^*$ when $n \to \infty$. Such a x^* is in fact a fixed point of T by

$$\begin{aligned} 0 &\leq d(x*, T(x^*)) \leq d(x^*, x_n) + d(x_n, T(x^*)) \\ &= d(x^*, x_n) + d(T(x_{n-1}), T(x^*)) \\ &\leq d(x^*, x_n) + \eta d(x_{n-1}, x^*) \to 0 \quad (\text{if } n \to \infty). \end{aligned}$$

Whence, $T(x^*) = x^*$. Now if there is another point $x_1^* \in X$ such that $T(x_1^*) = x_1^*$, by $1 < \eta < 1$ and

$$d(x^*, x_1^*) = d(T(x^*), T(x_1^*)) \leq \eta d(x^*, x_1^*)$$

There must be $d(x^*, x_1^*) = 0$, i.e., $x^* = x_1^*$. Thus such a fixed point x^* is unique. □

§1.6 SMARANDACHE MULTI-SPACES

1.6.1 Smarandache Multi-Space. Let Σ be a finite or infinite set. A *rule* or a *law* on a set Σ is a mapping $\underbrace{\Sigma\times\Sigma\cdots\times\Sigma}_{n}\rightarrow\Sigma$ for some integers n. Then a *mathematical space* is nothing but a pair $(\Sigma;\mathcal{R})$, where $\mathcal{R}$ consists those of rules on Σ by logic providing all these resultants are still in Σ.

Definition 1.6.1 *Let* $(\Sigma_1;\mathcal{R}_1)$ *and* $(\Sigma_2;\mathcal{R}_2)$ *be two mathematical spaces. If* $\Sigma_1\neq\Sigma_2$ *or* $\Sigma_1=\Sigma_2$ *but* $\mathcal{R}_1\neq\mathcal{R}$ *are said to be different, otherwise, identical.*

The Smarandache multi-space is a qualitative notion defined following.

Definition 1.6.2 *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be m mathematical spaces, different two by two. A Smarandache multi-space* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m}\Sigma_i$ *with rules* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *on* $\widetilde{\Sigma}$, *i.e., the rule* $\mathcal{R}_i$ *on* Σ_i *for integers* $1\leq i\leq m$, *denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

1.6.2 Multi-Space Type. By Definition 1.6.2, a Smarandache multi-space $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ is dependent on spaces $\Sigma_1,\Sigma_2,\cdots,\Sigma_m$ and rulers $\mathcal{R}_1,\mathcal{R}_2,\cdots,\mathcal{R}_m$. There are many types of Smarandache multi-spaces.

Definition 1.6.3 *A Smarandache multi-space* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *with* $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *is a finite if each* $\Sigma_i,\ 1\leq i\leq m$ *is finite, otherwise, infinite.*

Definition 1.6.4 *A Smarandache multi-space* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *with* $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *is a metric space if each* $(\Sigma_i;\mathcal{R}_i)$ *is a metric space, otherwise, a non-metric space.*

Definition 1.6.5 *A Smarandache multi-space* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *with* $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *is countable if each* $(\Sigma_i;\mathcal{R}_i)$ *is countable, otherwise, uncountable.*

1.6.3 Example. As we known, there are many kinds of spaces such as those of topological spaces, Euclidean spaces, metric spaces, $\cdots$ in classical mathematics and spacetimes in physics. All of them can be combined into a Smarandache multi-space $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$. We list some of these Smarandache multi-spaces following.

Example 1.6.1 Let $S_1,S_2,\cdots,S_m$ be m finite or infinite sets. By Definition 1.6.2, we get a multi-space $\widetilde{S}=\bigcup\limits_{i=1}^{m}S_i$. In fact, it is still a finite or infinite set.

Example 1.6.2 Let $T_1,T_2,\cdots,T_m$ be m partially order sets. By Definition 1.6.2, we get a partially order multi-space $\widetilde{T}=\bigcup\limits_{i=1}^{m}T_i$. In fact, it is also a partially order set.

Example 1.6.3 Let $(A_1; \circ_1), (A_2; \circ_2), \cdots, (A_m; \circ_m)$ be m finite or infinite algebraic systems such as those of groups, rings or fields. By Definition 1.6.2, we get an algebraic multi-space $\left(\widetilde{A}; O\right)$ with $\widetilde{A} = \bigcup\limits_{i=1}^{m} A_i$ and $O = \{\circ_i;\ 1 \leq i \leq m\}$. It maybe with different m closed operations.

Example 1.6.4 Let $M_1, M_2, \cdots, M_m$ be m vector spaces. By Definition 1.6.2, we get a vector multi-space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$. It may be a linear space or not.

Example 1.6.5 Let $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_m$ be m metric spaces. By Definition 1.6.2, we get a metric multi-space $\widetilde{\mathscr{T}} = \bigcup\limits_{i=1}^{m} \mathscr{T}_i$. It maybe with m different metrics.

Example 1.6.6 Let $\mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_m$ be m spacetimes. By Definition 1.6.2, we get a multi-spacetime $\widetilde{\mathscr{T}} = \bigcup\limits_{i=1}^{m} \mathscr{T}_i$. It maybe used to characterize particles in a parallel universe.

Example 1.6.7 Let $\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_m$ be m gravitational, electrostatic or electromagnetic field. By Definition 1.6.2, we get a multi-field $\widetilde{\mathscr{R}} = \bigcup\limits_{i=1}^{m} \mathscr{R}_i$. It contains partially gravitational or electrostatic fields, or partially electromagnetic fields.

§1.7 REMARKS

1.7.1 The multi-space and neutrosophic set were introduced by Smarandache in [Sma2] and then discussed himself in [Sma2]-[Sma5]. Indeed, the neutrosophic set is a simple way for measuring different degrees of spaces in a multi-space. Generally, we can define a function $\mu : \bigcup\limits_{i=1}^{n} S_i \rightarrow [0,1]$ with $\mu(S_i) \neq \mu(S_j)$ if $i \neq j$ for distinguishing each space S_i, $1 \leq i \leq n$. More conceptions appeared in Smarandache mathematics can be found in [Del1].

1.7.2 There are many standard textbooks on groups, rings, vector or metric spaces, such as those of [BiM1] and [NiD1] for modern algebra, [HoK1] for linear algebra, [Wan1], [Xum1] and [Rob1] for groups, [Xon1] for rings and [LiQ1] for metric spaces. The reference [BiM1] is an excellent textbook on modern algebra with first edition in 1941. The reader is inferred to these references [BiM1] and [NiD1] for topics discussed in this chapter, and then understand conceptions such as those of multi-group, multi-ring, multi-field, vector multi-space, metric multi-space, pseudo-Euclidean space and Smarandache geometry appeared in this book.

CHAPTER 2.

Graph Multi-Spaces

A graph G consisting of vertices and edges is itself a Smarandache multi-space, i.e., Smarandache multi-set if it is not an isolated vertex graph and vertices, edges distinct two by two, i.e., they are not equal in status in consideration. Whence, we are easily get two kinds of Smarandache multi-spaces by graphs. One consists of those of labeled graphs with order≥ 2 or bouquets B_n with $n \geq 1$. Another consists of those of graphs G possessing a graphical property $\mathscr{P}$ validated and invalided, or only invalided but in multiple distinct ways on G. For introducing such Smarandache multi-space, graphs and graph families, such as those of regular graphs, planar graphs and hamiltonian graphs are discussed in the first sections, including graphical sequences, eccentricity value sequences of graphs. Operations, i.e., these union, join and Cartesian product on graphs are introduced in Section 2.3 for finding multi-space representations of graphs. Then in Section 2.4, we show how to decompose a complete graph or a Cayley graph to typical graphs, i.e., a Smarandache multi-space consisting of these typical graphs. Section 2.5 concentrates on labeling symmetric graphs by Smarandache digital, Smarandache symmetric sequences and find symmetries both on graph structures and digits, i.e., beautiful geometrical figures with digits.

§2.1 GRAPHS

2.1.1 Graph. A *graph* G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \rightarrow V \times V$, where each element in V or E is a *label* on G. The sets V and E are called respectively the *vertex set* and *edge set* of G, denoted by $V(G)$ and $E(G)$.

An elements $v \in V(G)$ is *incident* with an element $e \in E(G)$ if $I(e) = (v, x)$ or (x, v) for an $x \in V(G)$. Usually, if $(u, v) = (v, u)$, denoted by uv or $vu \in E(G)$ for $\forall(u, v) \in E(G)$, then G is called to be a graph without orientation and abbreviated to *graph* for simplicity. Otherwise, it is called to be a directed graph with an orientation $u \rightarrow v$ on each edge (u, v). The cardinal numbers of $|V(G)|$ and $|E(G)|$ are called its *order* and *size* of a graph G, denoted by $|G|$ and $\varepsilon(G)$, respectively.

Let G be a graph. We can represent a graph G by locating each vertex u in G by a point $p(u)$, $p(u) \neq p(v)$ if $u \neq v$ and an edge (u, v) by a curve connecting points $p(u)$ and $p(v)$ on a plane $\mathbf{R}^2$, where $p : G \rightarrow p(G)$ is a mapping from the G to $\mathbf{R}^2$. For example, a graph $G = (V, E; I)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $I(e_i) = (v_i, v_i), 1 \leq i \leq 4; I(e_5) = (v_1, v_2) = (v_2, v_1), I(e_8) = (v_3, v_4) = (v_4, v_3), I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2), I(e_8) = I(e_9) = (v_4, v_1) = (v_1, v_4)$ can be drawn on a plane as shown in Fig.2.1.1.

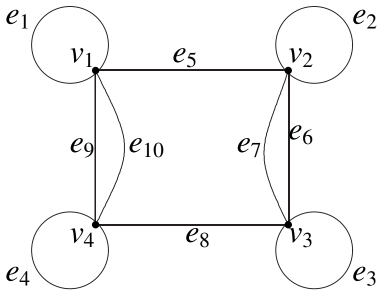


Fig. 2.1.1

In a graph $G = (V, E; I)$, for $\forall e \in E$, if $I(e) = (u, u), u \in V$, then e is called a *loop*. For $\forall e_1, e_2 \in E$, if $I(e_1) = I(e_2)$ and they are not loops, then e_1 and e_2 are called *multiple edges* of G. A graph is *simple* if it is loopless and without multiple edges, i.e., $\forall e_1, e_2 \in E(\Gamma)$, $I(e_1) \neq I(e_2)$ if $e_1 \neq e_2$ and for $\forall e \in E$, if $I(e) = (u, v)$, then $u \neq v$. In a simple graph, an edge (u, v) can be abbreviated to uv.

An edge $e \in E(G)$ can be divided into two semi-arcs e_u, e_v if $I(e) = (u, v)$. Call u the *root vertex* of the semi-arc e_u. Two semi-arc e_u, f_v are said to be *v-incident* or *e−incident* if $u = v$ or $e = f$. The set of all semi-arcs of a graph G is denoted by $X_{\frac{1}{2}}(G)$.

A *walk* in a graph Γ is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $e_i = (u_i, u_{i+1})$ for $1 \leq i \leq n$. The number n is the *length of the walk*. If $u_1 = u_{n+1}$, the walk is said to be *closed*, and *open* otherwise. For example, the sequence $v_1e_1v_1e_5v_2e_6v_3e_3v_3e_7v_2e_2v_2$ is a walk in Fig.2.1.1. A walk is a *trail* if all its edges are distinct and a *path* if all the vertices are distinct also. A closed path is usually called a *circuit* or *cycle*. For example, $v_1v_2v_3v_4$ and $v_1v_2v_3v_4v_1$ are respective path and circuit in Fig.2.1.1.

A graph $G = (V, E; I)$ is *connected* if there is a path connecting any two vertices in this graph. In a graph, a maximal connected subgraph is called a *component*. A graph G is *k-connected* if removing vertices less than k from G remains a connected graph.

A graph G is n-partite for an integer $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \cdots, V_n$ such that every edge joints a vertex of $V_i$to a vertex of V_j, $j \neq i$, $1 \leq i, j \leq n$. A *complete n-partite graph* G is such an n-partite graph with edges $uv \in E(G)$ for $\forall u \in V_i$ and $v \in V_j$ for $1 \leq i, j \leq n$, denoted by $K(p_1, p_2, \cdots, p_n)$ if $|V_i| = p_i$ for integers $1 \leq i \leq n$. Particularly, if $|V_i| = 1$ for integers $1 \leq i \leq n$, such a complete n-partite graph is called *complete graph* and denoted by K_n. In Fig.2.1.2, we can find the bipartite graph $K(4, 4)$ and the complete graph K_6. Usually, a complete subgraph of a graph is called a *clique*, and its a k-regular vertex-spanning subgraph also called a *k-factor*.

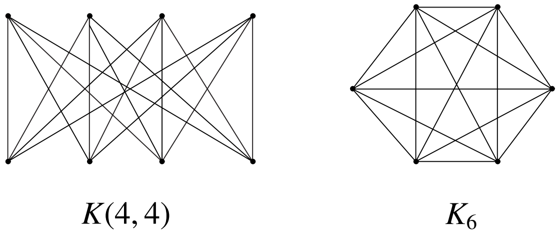


Fig.2.1.2

2.1.2 Isomorphic Graph. Let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs. They are *identical*, denoted by $G_1 = G_2$ if $V_1 = V_2, E_1 = E_2$ and $I_1 = I_2$. If there exists a $1-1$ mapping $\phi : E_1 \to E_2$ and $\phi : V_1 \to V_2$ such that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$ with the convention that $\phi(u, v) = (\phi(u), \phi(v))$, then we say that G_1 is *isomorphic* to G_2, denoted by $G_1 \simeq G_2$ and ϕ an *isomorphism* between G_1 and G_2. For simple graphs H_1, H_2, this definition can be simplified by $(u, v) \in I_1(E_1)$ if and only if $(\phi(u), \phi(v)) \in I_2(E_2)$ for $\forall u, v \in V_1$.

For example, let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs with

$$V_1 = \{v_1, v_2, v_3\}, \quad E_1 = \{e_1, e_2, e_3, e_4\},$$

$$I_1(e_1) = (v_1, v_2), I_1(e_2) = (v_2, v_3), I_1(e_3) = (v_3, v_1), I_1(e_4) = (v_1, v_1)$$

and

$$V_2 = \{u_1, u_2, u_3\}, \quad E_2 = \{f_1, f_2, f_3, f_4\},$$

$$I_2(f_1) = (u_1, u_2), I_2(f_2) = (u_2, u_3), I_2(f_3) = (u_3, u_1), I_2(f_4) = (u_2, u_2),$$

i.e., the graphs shown in Fig.2.1.3.

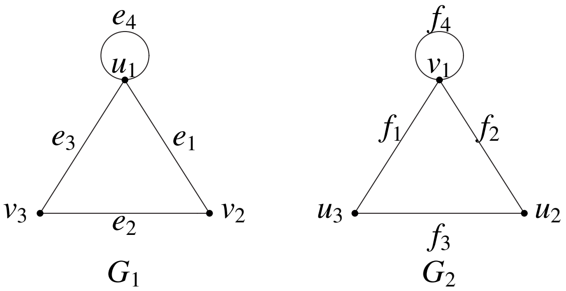


Fig. 2.1.3

Define a mapping $\phi : E_1 \bigcup V_1 \to E_2 \bigcup V_2$ by $\phi(e_1) = f_2, \phi(e_2) = f_3, \phi(e_3) = f_1, \phi(e_4) = f_4$ and $\phi(v_i) = u_i$ for $1 \leq i \leq 3$. It can be verified immediately that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$. Therefore, ϕ is an isomorphism between G_1 and G_2, i.e., G_1 and G_2 are isomorphic.

If $G_1 = G_2 = G$, an isomorphism between G_1 and G_2 is said to be an *automorphism* of G. All automorphisms of a graph G form a group under the composition operation, i.e., $\phi\theta(x) = \phi(\theta(x))$ for $x \in E(G) \bigcup V(G)$, denoted by Aut$G$.

2.1.3 Subgraph. A graph $H = (V_1, E_1; I_1)$ is a *subgraph* of a graph $G = (V, E; I)$ if $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \to V_1 \times V_1$. We use $H \prec G$ to denote that H is a subgraph of G. For example, graphs G_1, G_2, G_3 are subgraphs of the graph G in Fig.2.1.4.

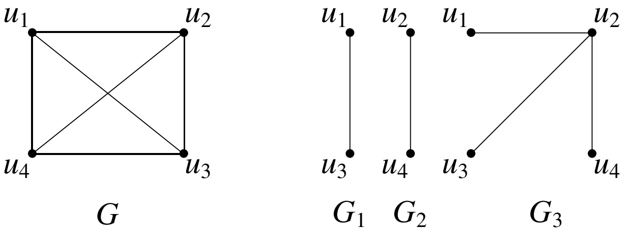


Fig. 2.1.4

For a nonempty subset U of the vertex set $V(G)$ of a graph G, the subgraph $\langle U\rangle$ of G *induced* by U is a graph having vertex set U and whose edge set consists of these edges of G incident with elements of U. A subgraph H of G is called *vertex-induced* if $H \simeq \langle U\rangle$ for some subset U of $V(G)$. Similarly, for a nonempty subset F of $E(G)$, the subgraph $\langle F\rangle$ induced by F in G is a graph having edge set F and whose vertex set consists of vertices of G incident with at least one edge of F. A subgraph H of G is *edge-induced* if $H \simeq \langle F\rangle$ for some subset F of $E(G)$. In Fig.2.1.4, subgraphs G_1 and G_2 are both vertex-induced subgraphs $\langle\{u_1, u_4\}\rangle$, $\langle\{u_2, u_3\}\rangle$ and edge-induced subgraphs $\langle\{(u_1, u_4)\}\rangle$, $\langle\{(u_2, u_3)\}\rangle$. For a subgraph H of G, if $|V(H)| = |V(G)|$, then H is called a *spanning subgraph* of G. In Fig.2.1.4, the subgraph G_3 is a spanning subgraph of the graph G.

A spanning subgraph without circuits is called a *spanning forest*. It is called a *spanning tree* if it is connected. A path is also a tree in which each vertex has valency 2 unless the two pendent vertices valency 1. We define the *length* of P_n to be $n - 1$. The following characteristic for spanning trees of a connected graph is well-known.

Theorem 2.1.1 *A subgraph T of a connected graph G is a spanning tree if and only if T is connected and $E(T) = |V(G)| - 1$.*

Proof The necessity is obvious. For its sufficiency, since T is connected and $E(T) = |V(G)| - 1$, there are no circuits in T. Whence, T is a spanning tree. □

2.1.4 Graphical Sequence. Let G be a graph. For $\forall u \in V(G)$, the neighborhood $N_G(u)$ of vertex u in G is defined by $N_G(u) = \{v|\forall(u, v) \in E(G)\}$. The cardinal number $|N_G(u)|$ is called the *valency of vertex u* in the graph G and denoted by $\rho_G(u)$. A vertex v with $\rho_G(v) = 0$ is called an *isolated vertex* and $\rho_G(v) = 1$ a *pendent vertex*. Now we arrange all vertices valency of G as a sequence $\rho_G(u) \geq \rho_G(v) \geq \cdots \geq \rho_G(w)$. Call this sequence the *valency sequence* of G. By enumerating edges in $E(G)$, the following result

$$\sum_{u\in V(G)} \rho_G(u) = 2|E(G)|$$

holds. Let $\rho_1, \rho_2, \cdots, \rho_p$ be a sequence of non-negative integers. If there exists a graph whose valency sequence is $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, we say that $\rho_1, \rho_2, \cdots, \rho_p$ is a *graphical sequence*. We know results following for graphical sequences.

Theorem 2.1.2(Havel,1955 and Hakimi, 1962) *A sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, $p \geq 2, \rho_1 \geq 1$ is graphical if and only if the sequence*

$\rho_2-1,\rho_3-1,\cdots,\rho_{\rho_1+1}-1,\rho_{\rho_1+2},\cdots,\rho_p$ is graphical.

Theorem 2.1.3(Erdös and Gallai, 1960) *A sequence $\rho_1,\rho_2,\cdots,\rho_p$ of non-negative integers with $\rho_1\geq\rho_2\geq\cdots\geq\rho_p$ is graphical if and only if $\sum\limits_{i=1}^{p}\rho_i$ is even and for each integer $n, 1\leq n\leq p-1$,*

$$\sum_{i=1}^{n}\rho_i\leq n(n-1)+\sum_{i=n+1}^{p}min\{n,\rho_i\}.$$

A graph G with vertex set $V(G)=\{v_1,v_2,\cdots,v_p\}$ and edge set $E(G)=\{e_1,e_2,\cdots,e_q\}$ can be also described by that of matrixes. One such a matrix is a $p\times q$ *adjacency matrix* $A(G)=[a_{ij}]_{p\times q}$, where $a_{ij}=|I^{-1}(v_i,v_j)|$. Thus, the adjacency matrix of a graph G is symmetric and is a $0,1$-matrix having 0 entries on its main diagonal if G is simple. For example, the adjacency matrix $A(G)$ of the graph in Fig.2.1.1 is

$$A(G)=\begin{bmatrix}1&1&0&2\\1&1&2&0\\0&2&1&1\\2&0&1&1\end{bmatrix}$$

2.1.5 Eccentricity Value Sequence. For a connected graph G, let $x,y\in V(G)$. The distance $d(x,y)$ from x to y in G is defined by

$$d_G(x,y)=min\{\ |V(P(x,y))|-1\ |\ P(x,y) \text{ is a path connecting x and } y\ \}$$

and the *eccentricity* $e_G(u)$ of for $u\in V(G)$ is defined by

$$e_G(u)=max\{\ d_G(u,x)\ |\ x\in V(G)\}.$$

A vertex u^+ is called an *ultimate vertex* of vertex u if $d(u,u^+)=e_G(u)$. Not loss of generality, arrange these eccentricities of vertices in G in order $e_G(v_1),e_G(v_2),\cdots,e_G(v_n)$ with $e_G(v_1)\leq e_G(v_2)\leq\cdots\leq e_G(v_n)$, where $\{v_1,v_2,\cdots,v_n\}=V(G)$. The sequence $\{e_G(v_i)\}_{1\leq i\leq s}$ is called the *eccentricity sequence* of G. If $\{e_1,e_2,\cdots,e_s\}=\{e_G(v_1),e_G(v_2),\cdots,e_G(v_n)\}$ and $e_1<e_2<\cdots<e_s$, the sequence $\{e_i\}_{1\leq i\leq s}$ is called the *eccentricity value sequence* of G. For convenience, we abbreviate an integer sequence $\{r-1+i\}_{1\leq i\leq s+1}$ to $[r,r+s]$.

The *radius* $r(G)$ and *diameter* $D(G)$ of graph G are respectively defined by $r(G)=\min\{e_G(u)|u\in V(G)\}$ and $D(G)=\max\{e_G(u)|u\in V(G)\}$. Particularly, if $r(G)=D(G)$, such a graph G is called to be a *self-centered graph*, i.e., its eccentricity value sequence is nothing but $[r(G), r(G)]$.

For $\forall x \in V(G)$, define a *distance decomposition* $\{V_i(x)\}_{1\leq i\leq e_G(x)}$ *of* G *in root* x by

$$G = V_1(x)\bigoplus V_2(x)\bigoplus\cdots\bigoplus V_{e_G(x)}(x),$$

where $V_i(x) = \{\ u\ |d(x,u) = i,\ u \in V(G)\}$ for any integer i, $0 \leq i \leq e_G(x)$. Then a necessary and sufficient condition for the eccentricity value sequence of simple graph is obtained in the following.

Theorem 2.1.4 *A non-decreasing integer sequence* $\{r_i\}_{1\leq i\leq s}$ *is a graphical eccentricity value sequence if and only if*

(1) $r_1 \leq r_s \leq 2r_1$;

(2) $\triangle(r_{i+1}, r_i) = |r_{i+1} - r_i| = 1$ *for any integer* i, $1 \leq i \leq s-1$.

Proof If there is a graph G whose eccentricity value sequence is $\{r_i\}_{1\leq i\leq s}$, then $r_1 \leq r_s$ is trivial. Now we choose three different vertices u_1, u_2, u_3 in G such that $e_G(u_1) = r_1$ and $d_G(u_2, u_3) = r_s$. By definition, we know that $d(u_1, u_2) \leq r_1$ and $d(u_1, u_3) \leq r_1$. According to the triangle inequality on distance, we know that $r_s = d(u_2, u_3) \leq d_G(u_2, u_1) + d_G(u_1, u_3) = d_G(u_1, u_2) + d_G(u_1, u_3) \leq 2r_1$. Thus $r_1 \leq r_s \leq 2r_1$.

Now if $\{e_i\}_{1\leq i\leq s}$ is the eccentricity value sequence of a graph G, define $\triangle(i) = e_{i+1} - e_i$, $1 \leq i \leq n-1$. We assert that $0 \leq \triangle(i) \leq 1$. If this assertion is not true, then there must exists a positive integer I, $1 \leq I \leq n-1$ such that $\triangle(I) = e_{I+1} - e_I \geq 2$. Choose a vertex $x \in V(G)$ such that $e_G(x) = e_I$ and consider the distance decomposition $\{V_i(x)\}_{0\leq i\leq e_G(x)}$ of G in root x.

Clearly, $e_G(x) - 1 \leq e_G(u_1) \leq e_G(x) + 1$ for any vertex $u_1 \in V_1(G)$. Since $\triangle(I) \geq 2$, there does not exist a vertex with the eccentricity $e_G(x)+1$. Whence, we get $e_G(u_1) \leq e_G(x)$ for $\forall u_1 \in V_1(x)$. Now if we have proved that $e_G(u_j) \leq e_G(x)$ for $\forall u_j \in V_j(x)$, $1 \leq j < e_G(x)$, we consider these eccentricity values of vertices in $V_{j+1}(x)$. Let $u_{j+1} \in V_{j+1}(x)$. According to the definition of $\{V_i(x)\}_{0\leq i\leq e_G(x)}$, there must exists a vertex $u_j \in V_j(x)$ such that $(u_j, u_{j+1}) \in E(G)$. Consider the distance decomposition $\{V_i(u_j)\}_{0\leq j\leq e_G(u)}$ of G in root u_j. Notice that $u_{j+1} \in V_1(u_j)$. Thereby we get that

$$e_G(u_{j+1}) \leq e_G(u_j) + 1 \leq e_G(x) + 1.$$

Because we have assumed that there are no vertices with the eccentricity $e_G(x) + 1$, so $e_G(u_{j+1}) \leq e_G(x)$ for any vertex $u_{j+1} \in V_{j+1}(x)$. Continuing this process, we know that

$e_G(y) \leq e_G(x) = e_I$ for any vertex $y \in V(G)$. But then there are no vertices with the eccentricity $e_I + 1$, contradicts to the assumption that $\triangle(I) \geq 2$. Therefore $0 \leq \triangle(i) \leq 1$ and $\triangle(r_{i+1}, r_i) = 1, 1 \leq i \leq s-1$.

For any integer sequence $\{r_i\}_{1\leq i\leq s}$ with conditions (i) and (ii) hold, it can be simply written as $\{r, r+1, \cdots, r+s-1\} = [r, r+s-1$, where $s \leq r$. We construct a graph with the eccentricity value sequence $[r, r+s-1]$ in the following.

Case 1. $s = 1$.

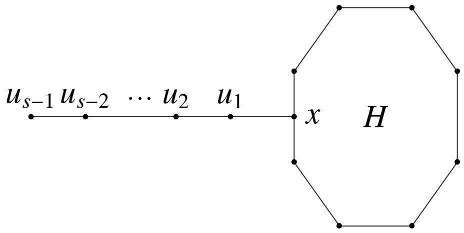
In this case, $\{r_i\}_{1\leq i\leq s} = [r, r]$. We can choose any self-centered graph with $r(G) = r$, for example, the circuit C_{2r}. Clearly, the eccentricity value sequence of C_{2r} is $[r, r]$.

Case 2. $s \geq 2$.

Choose a self-centered graph H with $r(H) = r,\ x \in V(H)$ and a path $P_s = u_0u_1 \cdots u_{s-1}$. Define a new graph $G = P_s \bigodot H$ as follows:

$$V(G) = V(P_s)\bigcup V(H) \setminus \{u_0\}, \qquad E(G) = E(P_s)\bigcup\{xu_1\}\bigcup E(H) \setminus \{u_1u_0\}$$

such as the graph G shown in Fig.2.1.5.



$G = P_s \bigodot H$

Fig 2.1.5

Then we know that $e_G(x) = r$, $e_G(u_{s-1}) = r+s-1$ and $r \leq e_G(x) \leq r+s-1$ for all other vertices $x \in V(G)$. Therefore, the eccentricity value sequence of G is $[r, r+s-1]$. This completes the proof. □

For a given eccentricity value l, the *multiplicity set* $N_G(l)$ is defined by $N_G(l) = \{\ x \mid x \in V(G), e(x) = l\ \}$. Jordan proved that the $\langle N_G(r(G))\rangle$ in a tree is a vertex or two adjacent vertices in 1869. For a general graph, maybe a tree, we get the following result which generalizes Jordan's result on trees.

Theorem 2.1.5 *Let* $\{r_i\}_{1\leq i\leq s}$ *be a graphical eccentricity value sequence. If* $|N_G(r_I)| = 1$, *then there must be* $I = 1$, *i.e.,* $|N_G(r_i)| \geq 2$ *for any integer* $i, 2 \leq i \leq s$.

Proof Let G be a graph with the eccentricity value sequence $\{r_i\}_{1\leq i\leq s}$ and $N_G(r_I) = \{x_0\}, e_G(x_0) = r_I$. We prove that $e_G(x) > e_G(x_0)$ for any vertex $x \in V(G) \setminus \{x_0\}$. Consider the distance decomposition $\{V_i(x_0)\}_{0\leq i\leq e_G(x_0)}$ of G in root x_0. First, we prove that $e_G(v_1) = e_G(x_0)+1$ for any vertex $v_1 \in V_1(x_0)$. Since $e_G(x_0)-1 \leq e_G(v_1) \leq e_G(x_0)+1$ for any vertex $v_1 \in V_1(x_0)$, we only need to prove that $e_G(v_1) > e_G(x_0)$ for any vertex $v_1 \in V_1(x_0)$. In fact, since for any ultimate vertex x_0^+ of x_0, we have $d_G(x_0, x_0^+) = e_G(x_0)$. So $e_G(x_0^+) \geq e_G(x_0)$. Notice that $N_G(e_G(x_0)) = \{x_0\}, x_0^+ \notin N_G(e_G(x_0))$. Consequently, $e_G(x_0^+) > e_G(x_0)$. Choose $v_1 \in V_1(x_0)$. Assume the shortest path from v_1 to x_0^+ is $P_1 = v_1v_2\cdots v_sx_0^+$ and $x_0 \notin V(P_1)$. Otherwise, we already have $e_G(v_1) > e_G(x_0)$. Now consider the distance decomposition $\{V_i(x_0^+)\}_{0\leq i\leq e_G(x_0^+)}$ of G in root x_0^+. We know that $v_s \in V_1(x_0^+)$. Thus we get that

$$e_G(x_0^+) - 1 \leq e_G(v_s) \leq e_G(x_0^+) + 1.$$

Therefore, $e_G(v_s) \geq e_G(x_0^+) - 1 \geq e_G(x_0)$. Because $N_G(e_G(x_0)) = \{x_0\}$, so $v_s \notin N_G(e_G(x_0))$. This fact enables us finally getting that $e_G(v_s) > e_G(x_0)$.

Similarly, choose $v_s, v_{s-1}, \cdots, v_2$ to be root vertices respectively and consider these distance decompositions of G in roots $v_s, v_{s-1}, \cdots, v_2$, we find that

$$\begin{aligned}&e_G(v_s) > e_G(x_0),\\ &e_G(v_{s-1}) > e_G(x_0),\\ &\cdots\cdots\cdots\cdots\cdots,\\ &e_G(v_1) > e_G(x_0).\end{aligned}$$

Therefore, $e_G(v_1) = e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$. Now consider these vertices in $V_2(x_0)$. For $\forall v_2 \in V_2(x_0)$, assume that v_2 is adjacent to $u_1, u_1 \in V_1(x_0)$. We know that $e_G(v_2) \geq e_G(u_1)-1 \geq e_G(x_0)$. Since $|N_G(e_G(x_0))| = |N_G(r_I)| = 1$, we get $e_G(v_2) \geq e_G(x_0)+1$.

Now if we have proved $e_G(v_k) \geq e_G(x_0) + 1$ for any vertex $v_k \in V_1(x_0) \bigcup V_2(x_0) \bigcup \cdots \bigcup V_k(x_0)$ for $1 \leq k < e_G(x_0)$. Let $v_{k+1} \in V_{k+1}(x_0)$ and assume that v_{k+1} is adjacent to u_k in $V_k(x_0)$. Then we know that $e_G(v_{k+1}) \geq e_G(u_k) - 1 \geq e_G(x_0)$. Since $|N_G(e_G(x_0))| = 1$, we get that $e_G(v_{k+1}) \geq e_G(x_0) + 1$. Therefore, $e_G(x) > e_G(x_0)$ for any vertex $x, x \in V(G) \setminus \{x_0\}$. Thus, if $|N_G(r_I)| = 1$, then there must be $I = 1$. □

Theorem 2.1.5 is the best possible in some cases of trees. For example, the eccentricity value sequence of a path P_{2r+1} is $[r, 2r]$ and we have that $|N_G(r)| = 1$ and $|N_G(k)| = 2$

for $r+1 \leq k \leq 2r$. But for graphs not being trees, we only found some examples satisfying $|N_G(r_1)| = 1$ and $|N_G(r_i)| > 2$. A non-tree graph with the eccentricity value sequence $[2, 3]$ and $|NG(2)| = 1$ can be found in Fig.2 in the reference [MaL2].

§2.2 GRAPH EXAMPLES

Some important classes of graphs are introduced in the following.

2.2.1 Bouquet and Dipole. In graphs, two simple cases is these graphs with one or two vertices, which are just bouquets or dipoles. A graph $B_n = (V_b, E_b; I_b)$ with $V_b = \{\ O\ \}$, $E_b = \{e_1, e_2, \cdots, e_n\}$ and $I_b(e_i) = (O, O)$ for any integer $i, 1 \leq i \leq n$ is called a *bouquet* of n edges. Similarly, a graph $D_{s.l.t} = (V_d, E_d; I_d)$ is called a *dipole* if $V_d = \{O_1, O_2\}$, $E_d = \{e_1, e_2, \cdots, e_s, e_{s+1}, \cdots, e_{s+l}, e_{s+l+1}, \cdots, e_{s+l+t}\}$ and

$$I_d(e_i) = \begin{cases} (O_1, O_1), & \text{if } 1 \leq i \leq s, \\ (O_1, O_2), & \text{if } s+1 \leq i \leq s+l, \\ (O_2, O_2), & \text{if } s+l+1 \leq i \leq s+l+t. \end{cases}$$

For example, B_3 and $D_{2,3,2}$ are shown in Fig.2.2.1.

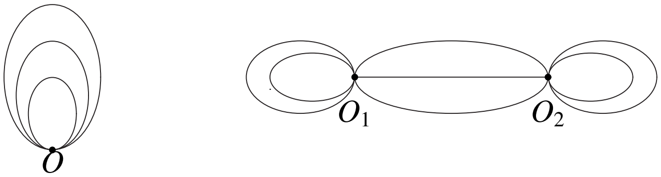


Fig. 2.2.1

In the past two decades, the behavior of bouquets on surfaces fascinated many mathematicians on topological graphs. Indeed, its behaviors on surfaces simplify the conception of surface. For such a contribution, a typical example is the classification theorem of surfaces. Thus by a combinatorial view, these connected sums of tori, or these connected sums of projective planes are nothing but a bouquet on surfaces.

2.2.2 Complete Graph. A *complete graph* $K_n = (V_c, E_c; I_c)$ is a simple graph with $V_c = \{v_1, v_2, \cdots, v_n\}$, $E_c = \{e_{ij}, 1 \leq i, j \leq n, i \neq j\}$ and $I_c(e_{ij}) = (v_i, v_j)$. Since K_n is simple, it can be also defined by a pair (V, E) with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_iv_j, 1 \leq i, j \leq n, i \neq j\}$.

The one edge graph K_2 and the triangle graph K_3 are both complete graphs.

A complete subgraph in a graph is called a *clique*. Obviously, every graph is a union of its cliques.

2.2.3 r-Partite Graph. A simple graph $G = (V, E; I)$ is *r-partite* for an integer $r \geq 1$ if it is possible to partition V into r subsets $V_1, V_2, \cdots, V_r$ such that for $\forall e \in E$, $I(e) = (v_i, v_j)$ for $v_i \in V_i$, $v_j \in V_j$ and $i \neq j, 1 \leq i, j \leq r$. Notice that by definition, there are no edges between vertices of V_i, $1 \leq i \leq r$. A vertex subset of this kind in a graph is called an *independent vertex subset*.

For $n = 2$, a 2-partite graph is also called a *bipartite*. It can be shown that *a graph is bipartite if and only if there are no odd circuits in this graph*. As a consequence, a tree or a forest is a bipartite graph since they are circuit-free.

Let $G = (V, E; I)$ be an r-partite graph and let $V_1, V_2, \cdots, V_r$ be its r-partite vertex subsets. If there is an edge $e_{ij} \in E$ for $\forall v_i \in V_i$ and $\forall v_j \in V_j$, where $1 \leq i, j \leq r, i \neq j$ such that $I(e) = (v_i, v_j)$, then we call G a *complete r-partite graph*, denoted by $G = K(|V_1|, |V_2|, \cdots, |V_r|)$. Whence, a complete graph is just a complete 1-partite graph. For an integer n, the complete bipartite graph $K(n, 1)$ is called a *star*. For a graph G, we have an obvious formula shown in the following, which corresponds to the neighborhood decomposition in topology.

$$E(G) = \bigcup_{x \in V(G)} E_G\left(x, N_G(x)\right).$$

2.2.4 Regular Graph. A graph G is *regular of valency k* if $\rho_G(u) = k$ for $\forall u \in V(G)$. These graphs are also called *k-regular*. There 3-regular graphs are referred to as *cubic graphs*. A k-regular vertex-spanning subgraph of a graph G is also called a k-factor of G.

For a k-regular graph G, by $k|V(G)| = 2|E(G)|$, thereby one of k and $|V(G)|$ must be an even number, i.e., there are no k-regular graphs of odd order with $k \equiv 1 (mod 2)$. A complete graph K_n is $(n-1)$-regular and a complete s-partite graph $K(p_1, p_2, \cdots, p_s)$ of order n with $p_1 = p_2 = \cdots = p_s = p$ is $(n-p)$-regular.

In regular graphs, those of simple graphs with high symmetry are particularly important to mathematics. They are related combinatorics with group theory and crystal geometry. We briefly introduce them in the following.

Let G be a simple graph and H a subgroup of Aut$G$. G is said to be *H-vertex transitive*, *H-edge transitive* or *H-symmetric* if H acts transitively on the vertex set $V(G)$, the edge set $E(G)$ or the set of ordered adjacent pairs of vertex of G. If $H = \mathrm{Aut}G$, an H-vertex transitive, an H-edge transitive or an H-symmetric graph is abbreviated to a *vertex-transitive*, an *edge-transitive* or a *symmetric* graph.

Now let Γ be a finite generated group and $S \subseteq \Gamma$ such that $1_\Gamma \notin S$ and $S^{-1} = \{x^{-1}|x \in S\} = S$. A *Cayley graph* $Cay(\Gamma : S)$ is a simple graph with vertex set $V(G) = \Gamma$ and edge set $E(G) = \{(g,h)|g^{-1}h \in S\}$. By the definition of Cayley graphs, we know that *a Cayley graph* $Cay(\Gamma : S)$ *is complete if and only if* $S = \Gamma \setminus \{1_\Gamma\}$ *and connected if and only if* $\Gamma = \langle S \rangle$.

Theorem 2.2.1 *A Cayley graph* $Cay(\Gamma : S)$ *is vertex-transitive.*

Proof For $\forall g \in \Gamma$, define a permutation ζ_g on $V(\mathrm{Cay}(\Gamma : S)) = \Gamma$ by $\zeta_g(h) = gh, h \in \Gamma$. Then ζ_g is an automorphism of $\mathrm{Cay}(\Gamma : S)$ for $(h,k) \in E(\mathrm{Cay}(\Gamma : S)) \Rightarrow h^{-1}k \in S \Rightarrow (gh)^{-1}(gk) \in S \Rightarrow (\zeta_g(h), \zeta_g(k)) \in E(\mathrm{Cay}(\Gamma : S))$.

Now we know that $\zeta_{kh^{-1}}(h) = (kh^{-1})h = k$ for $\forall h,k \in \Gamma$. Whence, $\mathrm{Cay}(\Gamma : S)$ is vertex-transitive. □

It should be noted that not every vertex-transitive graph is a Cayley graph of a finite group. For example, the Petersen graph is vertex-transitive but not a Cayley graph (see[CaM1], [GoR1 and [Yap1] for details). However, every vertex-transitive graph can be constructed almost like a Cayley graph. This result is due to Sabidussi in 1964. The readers can see [Yap1] for a complete proof of this result.

Theorem 2.2.2 *Let* G *be a vertex-transitive graph whose automorphism group is* A. *Let* $H = A_b$ *be the stabilizer of* $b \in V(G)$. *Then* G *is isomorphic with the group-coset graph* $C(A/H, S)$, *where* S *is the set of all automorphisms* x *of* G *such that* $(b, x(b)) \in E(G)$, $V(C(A/H,S)) = A/H$ *and* $E(C(A/H,S)) = \{(xH, yH)|x^{-1}y \in HSH\}$.

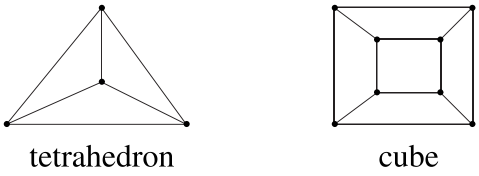


Fig. 2.2.2

2.2.5 Planar Graph. Every graph is drawn on the plane. A graph is *planar* if it can be drawn on the plane in such a way that edges are disjoint expect possibly for endpoints. When we remove vertices and edges of a planar graph G from the plane, each remained connected region is called a *face* of G. The length of the boundary of a face is called its *valency*. Two planar graphs are shown in Fig.2.2.2.

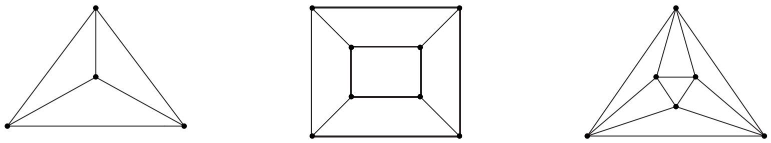
For a planar graph G, its order, size and number of faces are related by a well-known formula discovered by Euler.

Theorem 2.2.3 *let G be a planar graph with $\phi(G)$ faces. Then*

$$|G| - \varepsilon(G) + \phi(G) = 2.$$

Proof This result can be proved by induction on $\varepsilon(G)$. See [GrT1] or [MoT1] for a complete proof. □

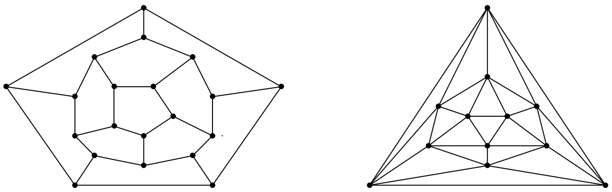
For an integer s, $s \geq 3$, an s-regular planar graph with the same length r for all faces is often called an (s, r)*-polyhedron*, which are completely classified by the ancient Greeks.



(3,3)
tetrahedron

(3,4)
hexahedron

(4,3)
octahedron



(3,5)
dodecahedron

(5,3)
icosahedron

Fig 2.2.3

Theorem 2.2.4 There are exactly five polyhedrons, two of them are shown in Fig.2.2.3.

Proof Let G be a k-regular planar graph with l faces. By definition, we know that $|G|k = \phi(G)l = 2\varepsilon(G)$. Whence, we get that $|G| = \dfrac{2\varepsilon(G)}{k}$ and $\phi(G) = \dfrac{2\varepsilon(G)}{l}$. According to Theorem 2.2.3, we get that

$$\frac{2\varepsilon(G)}{k} - \varepsilon(G) + \frac{2\varepsilon(G)}{l} = 2,$$

i.e.,

$$\varepsilon(G) = \frac{2}{\dfrac{2}{k} - 1 + \dfrac{2}{l}}.$$

Whence, $\frac{2}{k} + \frac{2}{l} - 1 > 0$. Since k, l are both integers and $k \geq 3,\ l \geq 3$, if $k \geq 6$, we get

$$\frac{2}{k} + \frac{2}{l} - 1 \leq \frac{2}{3} + \frac{2}{6} - 1 = 0,$$

contradicts to that $\frac{2}{k} + \frac{2}{l} - 1 > 0$. Therefore, $k \leq 5$. Similarly, $l \leq 5$. So we have $3 \leq k \leq 5$ and $3 \leq l \leq 5$. Calculation shows that all possibilities for (k, l) are $(k, l) = (3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$. The $(3, 3)$, $(3, 4)$, $(3, 5), (4, 3)$ and $(5, 3)$ polyhedrons are shown in Fig.2.2.3. □

An *elementary subdivision* on a graph G is a graph obtained from G replacing an edge $e = uv$ by a path uwv, where, $w \notin V(G)$. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivision. A graph H is defined to be a *homeomorphism of G* if either $H \simeq G$ or H is isomorphic to a subdivision of G. Kuratowski found the following characterization for planar graphs in 1930. For its a complete proof, see [BoM1] or [ChL1] for details.

Theorem 2.2.5 *A graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K(3, 3)$.*

2.2.6 Hamiltonian Graph. A graph G is *hamiltonian* if it has a circuit containing all vertices of G. Such a circuit is called a *hamiltonian circuit*. Similarly, if a path containing all vertices of a graph G, such a path is called a *hamiltonian path*.

For a given graph G and $V_1, V_2 \in V(G)$, define an *edge cut* $E_G(V_1, V_2)$ by

$$E_G(V_1, V_2) = \{\ (u, v) \in E(G) \mid u \in V_1, v \in V_2\}.$$

Then we have the following result for characterizing hamiltonian circuits.

Theorem 2.2.6 *A circuit C of a graph G without isolated vertices is a hamiltonian circuit if and only if for any edge cut $\mathcal{C}$, $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ and $|E(C) \bigcap E(\mathcal{C})| \geq 2$.*

Proof For any circuit C and an edge cut $\mathcal{C}$, the times crossing $\mathcal{C}$ as we travel along C must be even. Otherwise, we can not come back to the initial vertex. Whence, if C is a hamiltonian circuit, then $|E(C) \bigcap E(\mathcal{C})| \neq 0$. So $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$.

Conversely, if a circuit C satisfies $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$, we prove that C is a hamiltonian circuit of G. In fact, if $V(G) \setminus V(C) \neq$

$\emptyset$, choose $x \in V(G) \setminus V(C)$. Consider an edge cut $E_G(\{x\}, V(G) \setminus \{x\})$. Since $\rho_G(x) \neq 0$, we know that $|E_G(\{x\}, V(G) \setminus \{x\})| \geq 1$. But since $V(C) \bigcap (V(G) \setminus V(C)) = \emptyset$, there must be $|E_G(\{x\}, V(G) \setminus \{x\}) \bigcap E(C)| = 0$, contradicts to the fact that $|E(C) \bigcap E(C)| \geq 2$ for any edge cut C. Therefore $V(C) = V(G)$ and C is a hamiltonian circuit of G. $\square$

Let G be a simple graph. The *closure* of G, denoted by $C(G)$ is defined to be a graph obtained from G by recursively joining pairs of non-adjacent vertices whose valency sum is at least $|G|$ until no such pair remains. In 1976, Bondy and Chvátal proved a very useful theorem for hamiltonian graphs in [BoC1], seeing also [BoM1] following.

Theorem 2.2.7 *A simple graph is hamiltonian if and only if its closure is hamiltonian.*

This theorem generalizes Dirac's and Ore's theorems simultaneously following:

Dirac (1952): *Every connected simple graph G of order* $n \geq 3$ *with the minimum valency*$\geq \frac{n}{2}$ *is hamiltonian.*

Ore (1960): *If G is a simple graph of order* $n \geq 3$ *such that* $\rho_G(u) + \rho_G(v) \geq n$ *for all distinct non-adjacent vertices u and v, then G is hamiltonian.*

In 1984, Fan generalized Dirac's theorem to a localized form and proved that:

Let G be a 2-connected simple graph of order n. If the condition

$$\max\{\rho_G(u), \rho_G(v)\} \geq \frac{n}{2}$$

holds for $\forall u, v \in V(G)$ *provided* $d_G(u, v) = 2$*, then G is hamiltonian.*

After Fan's paper [Fan1], many researches concentrated on weakening Fan's condition and found new localized conditions for hamiltonian graphs. For example, the next result on hamiltonian graphs obtained by Shi in 1992 is such a result.

Theorem 2.2.8(Shi, 1992) *Let G be a 2-connected simple graph of order n. Then G contains a circuit passing through all vertices of valency*$\geq \frac{n}{2}$.

Proof Assume the assertion is false. Let $C = v_1 v_2 \cdots v_k v_1$ be a circuit containing as many vertices of valency$\geq \frac{n}{2}$ as possible and with an orientation on it. For $\forall v \in V(C)$, v^+ denotes the successor and v^- the predecessor of v on C. Set $R = V(G) \setminus V(C)$. Since G is 2-connected, there exists a path length than 2 connecting two vertices of C that is internally disjoint from C and containing one internal vertex x of valency$\geq \frac{n}{2}$ at least. Assume C and P are chosen in such a way that the length of P as small as possible. Let

$N_R(x) = N_G(x)\bigcap R$, $N_C(x) = N_G(x)\bigcap C$, $N_C^+(x) = \{v|v^- \in N_C(x)\}$ and $N_C^-(x) = \{v|v^+ \in N_C(x)\}$.

Not loss of generality, we assume $v_1 \in V(P)\bigcap V(C)$. Let v_t be the other vertex in $V(P)\bigcap V(C)$. By the way C was chosen, there exists a vertex v_s with $1 < s < t$ such that $\rho_G(v_s) \geq \frac{n}{2}$ and $\rho(v_i) < \frac{n}{2}$ for $1 < i < s$.

If $s \geq 3$, by the choice of C and P the sets

$$N_C^-(v_s) \setminus \{v_1\},\ N_C(x),\ N_R(v_s),\ N_R(x),\ \{x, v_{s-1}\}$$

are pairwise disjoint, which implies that

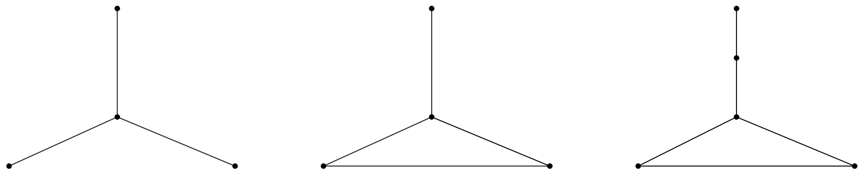
$$\begin{aligned} n &\geq |N_C^-(v_s) \setminus \{v_1\}| + |N_C(x)| + |N_R(v_s)| + |N_R(x)| + |\{x, v_{s-1}\}| \\ &= \rho_G(v_s) + \rho_G(x) + 1 \geq n + 1, \end{aligned}$$

a contradiction. If $s = 2$, then the sets

$$N_C^-(v_s),\ N_C(x),\ N_R(v_s),\ N_R(x),\ \{x\}$$

are pairwise disjoint, which yields a similar contradiction. □

There are three induced subgraphs shown in Fig.2.2.4, which are usually used for finding local conditions for hamiltonian graphs.



$K_{1.3}$ Z_1 Z_2

Fig 2.2.4

For an induced subgraph L of a simple graph G, a condition is called a *localized condition* $D_L(l)$ if $d_L(x, y) = l$ implies that $max\{\rho_G(x), \rho_G(y)\} \geq \frac{|G|}{2}$ for $\forall x, y \in V(L)$. Then we get the following result.

Theorem 2.2.9 *Let G be a 2-connected simple graph. If the localized condition $D_L(2)$ holds for induced subgraphs $L \simeq K_{1.3}$ or Z_2 in G, then G is hamiltonian.*

Proof By Theorem 2.2.8, we denote by $c_{\frac{n}{2}}(G)$ the maximum length of circuits passing through all vertices$\geq \frac{n}{2}$. Similar to the proof of Theorem 2.2.7, we know that for $x, y \in V(G)$, if $\rho_G(x) \geq \frac{n}{2}$, $\rho_G(y) \geq \frac{n}{2}$ and $xy \notin E(G)$, then $c_{\frac{n}{2}}(G \bigcup \{xy\}) = c_{\frac{n}{2}}(G)$. Otherwise, if $c_{\frac{n}{2}}(G \bigcup \{xy\}) > c_{\frac{n}{2}}(G)$, there exists a circuit of length $c_{\frac{n}{2}}(G \bigcup \{xy\})$ and passing through all vertices$\geq \frac{n}{2}$. Let $C_{\frac{n}{2}}$ be such a circuit and $C_{\frac{n}{2}} = xx_1x_2\cdots x_syx$ with $s = c_{\frac{n}{2}}(G \bigcup \{xy\}) - 2$. Notice that

$$N_G(x) \bigcap \left(V(G) \setminus V\left(C_{\frac{n}{2}}\left(G \bigcup \{xy\}\right)\right)\right) = \emptyset$$

and

$$N_G(y) \bigcap \left(V(G) \setminus V\left(C_{\frac{n}{2}}\left(G \bigcup \{xy\}\right)\right)\right) = \emptyset.$$

If there exists an integer i, $1 \leq i \leq s$, $xx_i \in E(G)$, then $x_{i-1}y \notin E(G)$. Otherwise, there is a circuit $C' = xx_ix_{i+1}\cdots x_syx_{i-1}x_{i-2}\cdots x$ in G passing through all vertices$\geq \frac{n}{2}$ with length $c_{\frac{n}{2}}(G \bigcup \{xy\})$, contradicts to the assumption that $c_{\frac{n}{2}}(G \bigcup \{xy\}) > c_{\frac{n}{2}}(G)$. Whence,

$$\rho_G(x) + \rho_G(y) \leq \left|V(G) \setminus V(C(C_{\frac{n}{2}}))\right| + \left|V(C(C_{\frac{n}{2}}))\right| - 1 = n - 1,$$

also contradicts to that $\rho_G(x) \geq \frac{n}{2}$ and $\rho_G(y) \geq \frac{n}{2}$. Therefore, $c_{\frac{n}{2}}(G \bigcup \{xy\}) = c_{\frac{n}{2}}(G)$ and generally, $c_{\frac{n}{2}}(C(G)) = c_{\frac{n}{2}}(G)$.

Now let C be a maximal circuit passing through all vertices$\geq \frac{n}{2}$ in the closure $C(G)$ of G with an orientation $\overrightarrow{C}$. According to Theorem 2.2.7, if $C(G)$ is non-hamiltonian, we can choose H be a component in $C(G) \setminus C$. Define $N_C(H) = (\bigcup\limits_{x \in H} N_{C(G)}(x)) \bigcap V(C)$. Since $C(G)$ is 2-connected, we get that $|N_C(H)| \geq 2$. This enables one to choose vertices $x_1, x_2 \in N_C(H)$, $x_1 \neq x_2$ and x_1 can arrive at x_2 along $\overrightarrow{C}$. Denote by $x_1\overrightarrow{C}x_2$ the path from x_1 to x_2 on $\overrightarrow{C}$ and $x_2\overleftarrow{C}x_1$ the reverse. Let P be a shortest path connecting x_1, x_2 in $C(G)$ and

$$u_1 \in N_{C(G)}(x_1) \bigcap V(H) \bigcap V(P), \quad u_2 \in N_{C(G)}(x_2) \bigcap V(H) \bigcap V(P).$$

Then

$$E(C(G)) \bigcap \left(\{x_1^-x_2^-, x_1^+x_2^+\} \bigcup E_{C(G)}\left(\{u_1, u_2\}, \{x_1^-, x_1^+, x_2^-, x_2^+\}\right)\right) = \emptyset$$

and $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \not\simeq K_{1.3}$ or $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \not\simeq K_{1.3}$. Otherwise, there exists a circuit longer than C, a contradiction. We need to consider two cases following.

Case 1. $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \not\simeq K_{1.3}$ and $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \not\simeq K_{1.3}$.

In this case, $x_1^-x_1^+ \in E(C(G))$ and $x_2^-x_2^+ \in E(C(G))$. By the maximality of C in $C(G)$, we have two claims.

Claim 1.1 $u_1 = u_2 = u$.

Otherwise, let $P = x_1u_1y_1\cdots y_lu_2$. By the choice of P, there must be

$$\langle\{x_1^-, x_1, x_1^+, u_1, y_1\}\rangle \simeq Z_2 \text{ and } \langle\{x_2^-, x_2, x_2^+, u_2, y_l\}\rangle \simeq Z_2$$

Since $C(G)$ also has the $D_L(2)$ property, we get that

$$\max\{\rho_{C(G)}(x_1^-), \rho_{C(G)}(u_1)\} \geq \frac{n}{2}, \ \max\{\rho_{C(G)}(x_12^-), \rho_{C(G)}(u_2)\} \geq \frac{n}{2}.$$

Whence, $\rho_{C(G)}(x_1^-) \geq \frac{n}{2}$, $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$ and $x_1^-x_2^- \in E(C(G))$, a contradiction.

Claim 1.2 $x_1x_2 \in E(C(G))$.

If $x_1x_2 \notin E(C(G))$, then $\left\langle\{x_1^-, x_1, x_1^+, u, x_2\}\right\rangle \simeq Z_2$. Otherwise, $x_2x_1^- \in E(C(G))$ or $x_2x_1^+ \in E(C(G))$. But then there is a circuit

$$C_1 = x_2^+\overrightarrow{C}x_1^-x_2ux_1\overrightarrow{C}x_2^-x_2^+ \text{ or } C_2 = x_2^+\overrightarrow{C}x_1ux_2x_1^+\overrightarrow{C}x_2^-x_2^+,$$

contradicts the maximality of C. Therefore, we know that

$$\langle\{x_1^-, x_1, x_1^+, u, x_2\}\rangle \simeq Z_2.$$

By the property $D_L(2)$, we get that $\rho_{C(G)}(x_1^-) \geq \frac{n}{2}$

Similarly, consider the induced subgraph $\left\langle\{x_2^-, x_2, x_2^+, u, x_2\}\right\rangle$, we get that $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$. Whence, $x_1^-x_2^- \in E(C(G))$, also a contradiction. Thereby we know the structure of G as shown in Fig.2.2.5.

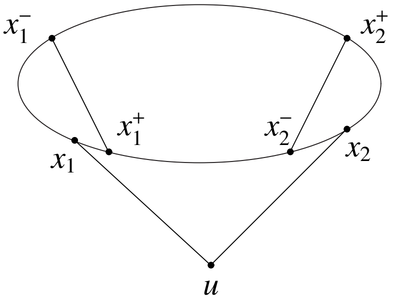


Fig 2.2.5

By the maximality of C in $C(G)$, it is obvious that $x_1^{--} \neq x_2^+$. We construct an induced subgraph sequence $\{G_i\}_{1\leq i\leq m}$, $m = |V(x_1^-\overleftarrow{C}x_2^+)|-2$ and prove there exists an integer $r, 1 \leq r \leq m$ such that $G_r \simeq Z_2$.

First, we consider the induced subgraph $G_1 = \left\langle\{x_1, u, x_2, x_1^-, x_1^{--}\}\right\rangle$. If $G_1 \simeq Z_2$, take $r = 1$. Otherwise, there must be

$$\{x_1^- x_2, x_1^{---} x_2, x_1^{---} u, x_1^{---} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $x_1^- x_2 \in E(C(G))$, or $x_1^{---} x_2 \in E(C(G))$, or $x_1^{---} u \in E(C(G))$, there is a circuit $C_3 = x_1^- \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1 u x_2 x_1^-$, or $C_4 = x_1^{--} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- x_1 u x_2 x_1^{---}$, or $C_5 = x_1^{--} \overleftarrow{C} x_1^+ x_1^- x_1 u x_1^{---}$. Each of these circuits contradicts the maximality of C. Therefore, $x_1^{---} x_1 \in E(C(G))$.

Now let $x_1^- \overleftarrow{C} x_2^+ = x_1^- y_1 y_2 \cdots y_m x_2^+$, where $y_0 = x_1^-, y_1 = x_1^{---}$ and $y_m = x_2^{++}$. If we have defined an induced subgraph G_k for any integer k and have gotten $y_i x_1 \in E(C(G))$ for any integer $i, 1 \leq i \leq k$ and $y_{k+1} \neq x_2^{++}$, then we define

$$G_{k+1} = \langle\{y_{k+1}, y_k, x_1, x_2, u\}\rangle .$$

If $G_{k+1} \simeq Z_2$, then $r = k + 1$. Otherwise, there must be

$$\{y_k u, y_k x_2, y_{k+1} u, y_{k+1} x_2, y_{k+1} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $y_k u \in E(C(G))$, or $y_k x_2 \in E(C(G))$, or $y_{k+1} u \in E(C(G))$, or $y_{k+1} x_2 \in E(C(G))$, there is a circuit $C_6 = y_k \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_{k-1} x_1 u y_k$, or $C_7 = y_k \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C}\ y_{k-1} x_1 u x_2 y_k$, or $C_8 = y_{k+1} \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_k x_1 u y_{k+1}$, or $C_9 = y_{k+1} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_k x_1 u\ x_2 y_{k+1}$. Each of these circuits contradicts the maximality of C. Thereby, $y_{k+1} x_1 \in E(C(G))$.

Continue this process. If there are no subgraphs in $\{G_i\}_{1\leq i\leq m}$ isomorphic to Z_2, we finally get $x_1 x_2^{++} \in E(C(G))$. But then there is a circuit $C_{10} = x_1^- \overleftarrow{C} x_2^{++} x_1 u x_2 x_2^+\ \overleftarrow{C} x_1^+ x_1^-$ in $C(G)$. Also contradicts the maximality of C in $C(G)$. Therefore, there must be an integer $r, 1 \leq r \leq m$ such that $G_r \simeq Z_2$.

Similarly, let $x_2^- \overleftarrow{C} x_1^+ = x_2^- z_1 z_2 \cdots z_t x_1^-$, where $t = |V(x_2^- \overleftarrow{C} x_1^+)| - 2$, $z_0 = x_2^-, z_1^{++} = x_2, z_t = x_1^{++}$. We can also construct an induced subgraph sequence $\{G^i\}_{1\leq i\leq t}$ and know that there exists an integer $h, 1 \leq h \leq t$ such that $G^h \simeq Z_2$ and $x_2 z_i \in E(C(G))$ for $0 \leq i \leq h-1$.

Since the localized condition $D_L(2)$ holds for an induced subgraph Z_2 in $C(G)$, we get that $\max\{\rho_{C(G)}(u), \rho_{C(G)}(y_{r-1})\} \geq \dfrac{n}{2}$ and $\max\{\rho_{C(G)}(u), \rho_{C(G)}(z_{h-1})\} \geq \dfrac{n}{2}$. Whence $\rho_{C(G)}(y_{r-1}) \geq \dfrac{n}{2}, \rho_{C(G)}(z_{h-1}) \geq \dfrac{n}{2}$ and $y_{r-1} z_{h-1} \in E(C(G))$. But then there is a circuit

$$C_{11} = y_{r-1} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} z_{h-2} x_2 u x_1 y_{r-2} \overrightarrow{C} x_1^- x_1^+ \overrightarrow{C} z_{h-1} y_{r-1}$$

in $C(G)$, where if $h = 1$, or $r = 1$, $x_2^- \overleftarrow{C} z_{h-2} = \emptyset$, or $y_{r-2} \overrightarrow{C} x_1^- = \emptyset$. Also contradicts the maximality of C in $C(G)$.

Case 2. $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \not\simeq K_{1.3}$, $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \simeq K_{1.3}$ or $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \simeq K_{1.3}$, but $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \not\simeq K_{1.3}$

Not loss of generality, we assume that

$$\langle\{x_1^-, x_1, x_1^+, u_1\}\rangle \not\simeq K_{1.3}, \quad \langle\{x_2^-, x_2, x_2^+, u_2\}\rangle \simeq K_{1.3}.$$

Since each induced subgraph $K_{1.3}$ in $C(G)$ possesses $D_L(2)$, we get that $\max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^-)\} \geq \frac{n}{2}$ and $\max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^+)\} \geq \frac{n}{2}$. Whence $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$, $\rho_{C(G)}(x_2^+) \geq \frac{n}{2}$ and $x_2^- x_2^+ \in E(C(G))$. Therefore, the discussion of Case 1 also holds in this case and yields similar contradictions.

Combining Case 1 with Case 2, the proof is complete. $\square$

Let G, $F_1, F_2, \cdots, F_k$ be $k+1$ graphs. If there are no induced subgraphs of G isomorphic to $F_i, 1 \leq i \leq k$, then G is called $\{F_1, F_2, \cdots, F_k\}$-*free*. We immediately get a consequence by Theorem 2.2.9.

Corollary 2.2.1 *Every* 2*-connected* $\{K_{1.3}, Z_2\}$*-free graph is hamiltonian.*

For a graph G, $u \in V(G)$ with $\rho_G(u) = l$, let H be a graph with l pendent vertices $v_1, v_2, \cdots, v_l$. Define a splitting operator $\vartheta : G \to G^{\vartheta(u)}$ on u by

$$\begin{aligned} V(G^{\vartheta(u)}) &= (V(G) \setminus \{u\}) \bigcup (V(H) \setminus \{v_1, v_2, \cdots, v_l\}), \\ E(G^{\vartheta(u)}) &= (E(G) \setminus \{ux_i \in E(G), 1 \leq i \leq l\}) \\ &\qquad \bigcup (E(H) \setminus \{v_i y_i \in E(H), 1 \leq i \leq l\}) \bigcup \{x_i y_i, 1 \leq i \leq l\}. \end{aligned}$$

Such number l is called the *degree of the splitting operator* ϑ and $N(\vartheta(u)) = H \setminus \{x_i y_i, 1 \leq i \leq l\}$ the *nucleus of* ϑ . A splitting operator is shown in Fig.2.2.6.

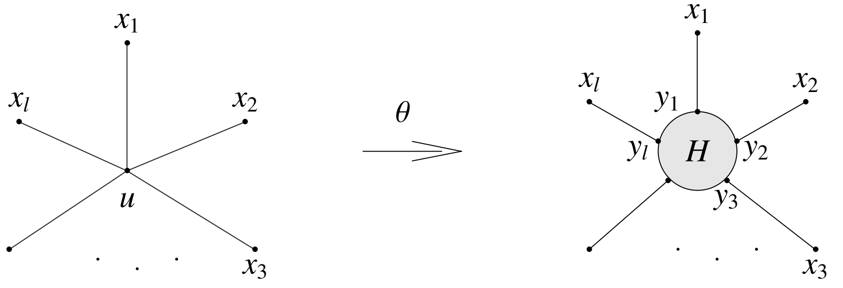


Fig 2.2.6

Erdös and Rényi raised a question in 1961: *in what model of random graphs is it true that almost every graph is hamiltonian?* Pósa and Korshuuov proved independently that for some constant c almost every labeled graph with n vertices and at least $n\log n$ edges is hamiltonian in 1974. Contrasting this probabilistic result, there is another property for hamiltonian graphs, i.e., there is a splitting operator ϑ such that $G^{\vartheta(u)}$ is non-hamiltonian for $\forall u \in V(G)$ of a graph G.

Theorem 2.2.10 *Let G be a graph. For $\forall u \in V(G)$, $\rho_G(u) = d$, there exists a splitting operator ϑ of degree d on u such that $G^{\vartheta(u)}$ is non-hamiltonian.*

Proof For any positive integer i, define a simple graph Θ_i by $V(\Theta_i) = \{x_i, y_i, z_i, u_i\}$ and $E(\Theta_i) = \{x_iy_i, x_iz_i, y_iz_i, y_iu_i, z_iu_i\}$. For integers $i, j \geq 1$, the point product $\Theta_i \odot \Theta_j$ of Θ_i and Θ_j is defined by

$$V(\Theta_i \odot \Theta_j) = V(\Theta_i)\bigcup V(\Theta_j) \setminus \{u_j\},$$
$$E(\Theta_i \odot \Theta_j) = E(\Theta_i)\bigcup E(\Theta_j)\bigcup\{x_iy_j, x_iz_j\} \setminus \{x_jy_j, x_jz_j\}.$$

Now let H_d be a simple graph with

$$V(H_d) = V(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1})\bigcup\{v_1, v_2, \cdots, v_d\},$$
$$E(H_d) = E(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1})\bigcup\{v_1u_1, v_2u_2, \cdots, v_du_d\}.$$

Then $\vartheta : G \rightarrow G^{\vartheta(w)}$ is a splitting operator of degree d as shown in Fig.2.2.7.

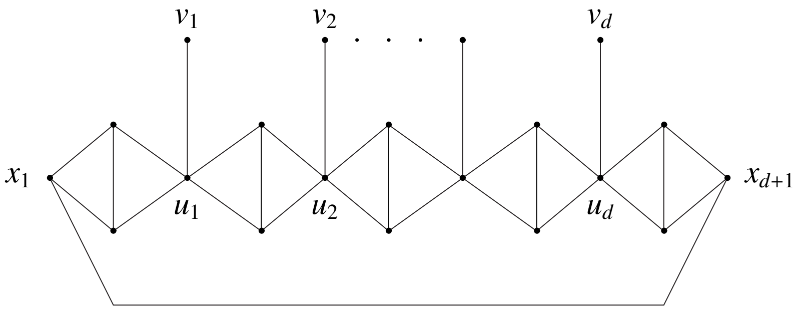


Fig 2.2.7

For any graph G and $w \in V(G), \rho_G(w) = d$, we prove that $G^{\vartheta(w)}$ is non-hamiltonian. In fact, If $G^{\vartheta(w)}$ is a hamiltonian graph, then there must be a hamiltonian path $P(u_i, u_j)$ connecting two vertices u_i, u_j for some integers $i, j, 1 \leq i, j \leq d$ in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$. However, there are no hamiltonian path connecting vertices u_i, u_j in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$ for any integer $i, j, 1 \leq i, j \leq d$. Therefore, $G^{\vartheta(w)}$ is non-hamiltonian. □

§2.3 GRAPH OPERATIONS WITH SEMI-ARC AUTOMORPHISMS

For two given graphs $G_1 = (V_1.E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$, there are a number of ways to produce new graphs from G_1 and G_2. Some of them are introduced in the following.

2.3.1 Union. A *union* $G_1 \bigcup G_2$ of graphs G_1 with G_2 is defined by

$$V(G_1 \bigcup G_2) = V_1 \bigcup V_2,\ E(G_1 \bigcup G_2) = E_1 \bigcup E_2,\ I(E_1 \bigcup E_2) = I_1(E_1) \bigcup I_2(E_2).$$

A graph consists of k disjoint copies of a graph H, $k \geq 1$ is denoted by $G = kH$. As an example, we find that

$$K_6 = \bigcup_{i=1}^{5} S_{1.i}$$

for graphs shown in Fig.2.3.1 following

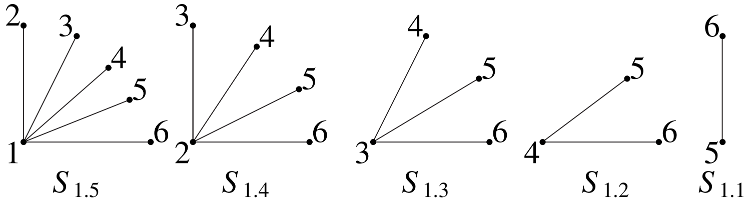


Fig. 2.3.1

and generally, $K_n = \bigcup\limits_{i=1}^{n-1} S_{1.i}$. Notice that kG is a multigraph with edge multiple k for any integer $k, k \geq 2$ and a simple graph G.

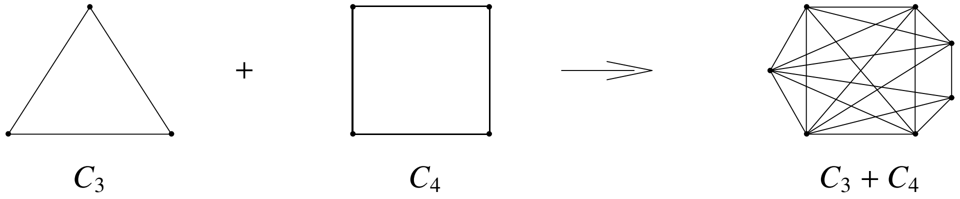


Fig 2.3.2

2.3.2 Complement and Join. A *complement* $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ such that vertices are adjacent in $\overline{G}$ if and only if these are not adjacent in G. A *join* $G_1 + G_2$ of G_1 with G_2 is defined by

$$V(G_1 + G_2) = V(G_1) \bigcup V(G_2),$$

$$E(G_1+G_2)=E(G_1)\bigcup E(G_2)\bigcup\{(u,v)|u\in V(G_1),v\in V(G_2)\}$$

$$I(G_1+G_2)=I(G_1)\bigcup I(G_2)\bigcup\{I(u,v)=(u,v)|u\in V(G_1),v\in V(G_2)\}.$$

Applying the join operation, we know that $K(m,n)\simeq\overline{K_m}+\overline{K_n}$. The join graph of circuits C_3 and C_4 is given in Fig.2.3.2.

2.3.3 Cartesian Product. A *Cartesian product* $G_1\times G_2$ of graphs G_1 with G_2 is defined by $V(G_1\times G_2)=V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of $G_1\times G_2$ are adjacent if and only if either $u_1=v_1$ and $(u_2,v_2)\in E(G_2)$ or $u_2=v_2$ and $(u_1,v_1)\in E(G_1)$. For example, $K_2\times P_6$ is shown in Fig.2.3.3 following.

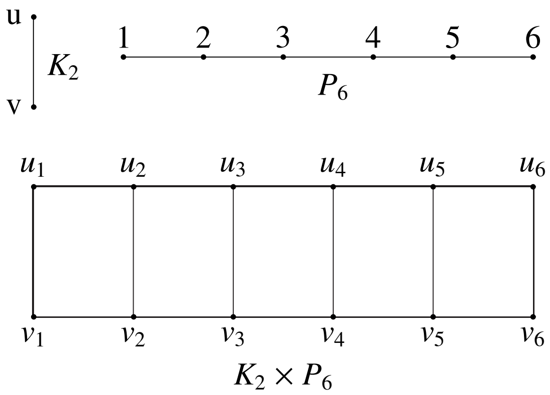


Fig.2.3.3

2.3.4 Semi-Arc Automorphism. For a simple graph G with n vertices, it is easy to verify that Aut$G\le S_n$, the symmetry group action on these n vertices of G.

G	Aut$G$	order
P_n	Z_2	2
C_n	D_n	$2n$
K_n	S_n	$n!$
$K_{m,n}(m\neq n)$	$S_m\times S_n$	$m!n!$
$K_{n,n}$	$S_2[S_n]$	$2n!^2$

Table 2.3.1

But in general, the situation is more complex. In Table 2.3.1, we present automorphism groups of some graphs. For generalizing the conception of automorphism, the semi-arc automorphism of a graph is introduced in the following.

Definition 2.3.1 *A one-to-one mapping ξ on $X_{\frac{1}{2}}(G)$ is called a semi-arc automorphism of a graph G if $\xi(e_u)$ and $\xi(f_v)$ are $v-$incident or $e-$incident if e_u and f_v are $v-$incident or $e-$incident for $\forall e_u, f_v \in X_{\frac{1}{2}}(G)$.*

All semi-arc automorphisms of a graph also form a group, denoted by $\mathrm{Aut}_{\frac{1}{2}}G$.The Table 2.3.2 following lists semi-arc automorphism groups of a few well-known graphs.

G	$\mathrm{Aut}_{\frac{1}{2}}G$	order
K_n	S_n	$n!$
$K_{n,n}$	$S_2[S_n]$	$2n!^2$
B_n	$S_n[S_2]$	$2^n n!$
$D_{0.n.0}$	$S_2 \times S_n$	$2n!$
$D_{n.k.l}(k \neq l)$	$S_2[S_k] \times S_n \times S_2[S_l]$	$2^{k+l} n!k!l!$
$D_{n.k.k}$	$S_2 \times S_n \times (S_2[S_k])^2$	$2^{2k+1} n!k!^2$

Table 2.3.2

In this table, $D_{0.n.0}$ is a dipole graph with 2 vertices, n multiple edges and $D_{n.k.l}$ is a generalized dipole graph with 2 vertices, n multiple edges, and one vertex with k bouquets and another, l bouquets. This table also enables us to find some useful information for semi-arc automorphism groups. For example, $\mathrm{Aut}_{\frac{1}{2}}K_n = \mathrm{Aut}K_n = S_n$, $\mathrm{Aut}_{\frac{1}{2}}B_n = S_n[S_2]$ but $\mathrm{Aut}B_n = S_n$, i.e., $\mathrm{Aut}_{\frac{1}{2}}B_n \neq \mathrm{Aut}B_n$ for any integer $n \geq 1$.

For $\forall g \in \mathrm{Aut}G$, there is an induced action $g|^{\frac{1}{2}} : X_{\frac{1}{2}}(G) \to X_{\frac{1}{2}}(G)$ defined by

$$\forall e_u \in X_{\frac{1}{2}}(G), \qquad g(e_u) = g(e)_{g(u)}.$$

All such induced actions on $X_{\frac{1}{2}}(G)$ by elements in $\mathrm{Aut}G$ are denoted by $\mathrm{Aut}G|^{\frac{1}{2}}$.

The graph B_n shows that $\mathrm{Aut}_{\frac{1}{2}}G$ may be not the same as $\mathrm{Aut}G|^{\frac{1}{2}}$. However, we get a result in the following.

Theorem 2.3.1 *For a graph G without loops,*

$$\mathrm{Aut}_{\frac{1}{2}}G = \mathrm{Aut}G|^{\frac{1}{2}}.$$

Proof By the definition, we only need to prove that for $\forall \xi_{\frac{1}{2}} \in \mathrm{Aut}_{\frac{1}{2}}G$, $\xi = \xi_{\frac{1}{2}}|_G \in \mathrm{Aut}G$ and $\xi_{\frac{1}{2}} = \xi|^{\frac{1}{2}}$. In fact, Let $e_u^{\circ}, f_x^{\bullet} \in X_{\frac{1}{2}}(G)$ with $\circ, \bullet \in \{+, -\}$, where $e = uv \in E(G)$, $f = xy \in E(G)$. Now if

$$\xi_{\frac{1}{2}}(e_u^{\circ}) = f_x^{\bullet},$$

by definition, we know that $\xi_{\frac{1}{2}}(e_v^{\circ}) = f_y^{\bullet}$. Whence, $\xi_{\frac{1}{2}}(e) = f$. That is, $\xi_{\frac{1}{2}}|_G \in \mathrm{Aut}G$.

By assumption, there are no loops in G. Whence, we know that $\xi_{\frac{1}{2}}|_G = 1_{\mathrm{Aut}G}$ if and only if $\xi_{\frac{1}{2}} = 1_{\mathrm{Aut}_{\frac{1}{2}}G}$. So $\xi_{\frac{1}{2}}$ is induced by $\xi_{\frac{1}{2}}|_G$ on $X_{\frac{1}{2}}(G)$. Thus,

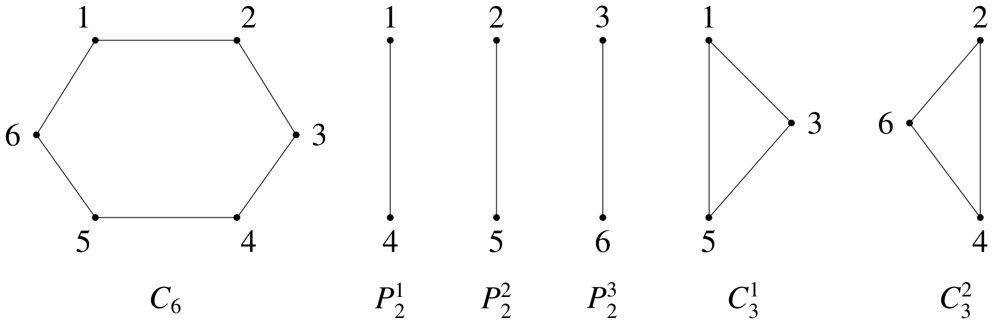
$$\mathrm{Aut}_{\frac{1}{2}}G = \mathrm{Aut}G|^{\frac{1}{2}}. \qquad \square$$

We have know that $\mathrm{Aut}_{\frac{1}{2}}B_n \neq \mathrm{Aut}B_n$ for any integer $n \geq 1$. Combining this fact with Theorem 2.1.3, we know the following.

Theorem 2.3.2 $\mathrm{Aut}_{\frac{1}{2}}G = \mathrm{Aut}G|^{\frac{1}{2}}$ *if and only if G is a loopless graph.*

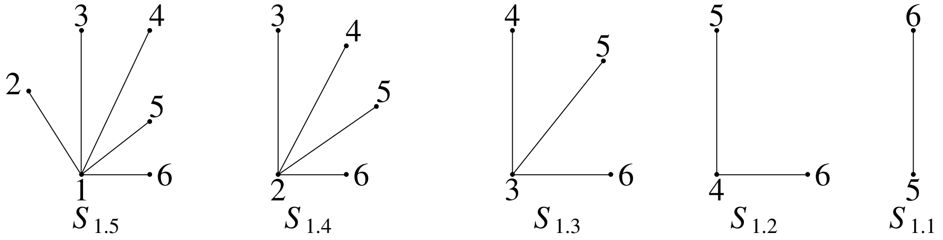
§2.4 DECOMPOSITIONS

2.4.1 Decomposition. A graph G can be really represented as a graph multi-space by decomposing it into subgraphs. for example, the complete graph K_6 with vertex set $\{1, 2, 3, 4, 5, 6\}$ has two families of subgraphs $\{C_6, C_3^1, C_3^2, P_2^1, P_2^2, P_2^3\}$ and $\{S_{1.5}, S_{1.4}, S_{1.3}, S_{1.2}, S_{1.1}\}$, such as those shown in Fig.2.4.1 and Fig.2.4.2.



C_6 P_2^1 P_2^2 P_2^3 C_3^1 C_3^2

Fig 2.4.1



$S_{1.5}$ $S_{1.4}$ $S_{1.3}$ $S_{1.2}$ $S_{1.1}$

Fig 2.4.2

Whence, we know that

$$E(K_6) = E(C_6)\bigcup E(C_3^1)\bigcup E(C_3^2)\bigcup E(P_2^1)\bigcup E(P_2^2)\bigcup E(P_2^3),$$
$$E(K_6) = E(S_{1.5})\bigcup E(S_{1.4})\bigcup E(S_{1.3})\bigcup E(S_{1.2})\bigcup E(S_{1.1}).$$

These formulae imply the conception of decomposition of graphs.

Generally, let G be a graph. A *decomposition* of G is a collection $\{H_i\}_{1\leq i\leq s}$ of subgraphs of G such that for any integer i, $1 \leq i \leq s$, $H_i = \langle E_i\rangle$ for some subsets E_i of $E(G)$ and $\{E_i\}_{1\leq i\leq s}$ is a partition of $E(G)$, denoted by

$$G = H_1\bigoplus H_2\bigoplus\cdots\bigoplus H_s.$$

By definition, we easily get decompositions for some well-known graphs such as

$$B_n = \bigcup_{i=1}^{n} B_1(O), \quad D_{k,m,n} = (\bigcup_{i=1}^{k} B_1(O_1))\bigcup(\bigcup_{i=1}^{m} K_2)\bigcup(\bigcup_{i=1}^{n} B_1(O_2)),$$

where $V(B_1)(O_1) = \{O_1\}, V(B_1)(O_2) = \{O_2\}$ and $V(K_2) = \{O_1, O_2\}$. The following result is obvious.

Theorem 2.4.1 *Any graph G can be decomposed to bouquets and dipoles, in where K_2 is seen as a dipole $D_{0.1.0}$.*

Theorem 2.4.2 *For every positive integer n, the complete graph K_{2n+1} can be decomposed to n hamiltonian circuits.*

Proof For $n = 1$, K_3 is just a hamiltonian circuit. Now let $n \geq 2$ and $V(K_{2n+1}) = \{v_0, v_1, v_2, \cdots, v_{2n}\}$. Arrange these vertices $v_1, v_2, \cdots, v_{2n}$ on vertices of a regular $2n$-gon and place v_0 in a convenient position not in the $2n$-gon. For $i = 1, 2, \cdots, n$, we define the edge set of H_i to be consisted of v_0v_i, v_0v_{n+i} and edges parallel to v_iv_{i+1} or edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2n$. Then we get that

$$K_{2n+1} = H_1\bigoplus H_2\bigoplus\cdots\bigoplus H_n$$

with each H_i, $1 \leq i \leq n$ being a hamiltonian circuit

$$v_0v_iv_{i+1}v_{i-1}v_{i+1}v_{i-2}\cdots v_{n+i-1}v_{n+i+1}v_{n+i}v_0.$$ □

Theorem 2.4.2 implies that $K_{2n+1} = \bigcup\limits_{i=1}^{n} H_i$ with

$$H_i = v_0v_iv_{i+1}v_{i-1}v_{i+1}v_{i-2}\cdots v_{n+i-1}v_{n+i+1}v_{n+i}v_0.$$

2.4.2 Factorization of Cayley Graph. Generally, every Cayley graph of a finite group Γ can be decomposed into 1-factors or 2-factors in a natural way shown in the following.

Theorem 2.4.3 *Let G be a vertex-transitive graph and let H be a regular subgroup of* Aut$G$. *Then for any chosen vertex $x, x \in V(G)$, there is a factorization*

$$G = \left(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H \right) \bigoplus \left(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H \right),$$

for G such that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and a 1-factor if $|H_{(x,y)}| = 2$.

Proof We prove the following claims.

Claim 1. $\forall x \in V(G), x^H = V(G)$ *and* $H_x = 1_H$.

Claim 2. *For* $\forall (x,y), (u,w) \in E(G)$, $(x,y)^H \bigcap (u,w)^H = \emptyset$ *or* $(x,y)^H = (u,w)^H$.

Claims 1 and 2 are holden by definition.

Claim 3. *For* $\forall (x,y) \in E(G), |H_{(x,y)}| = 1 \quad or \quad 2$.

Assume that $|H_{(x,y)}| \neq 1$. Since we know that $(x,y)^h = (x,y)$, i.e., $(x^h, y^h) = (x,y)$ for any element $h \in H_{(x,y)}$. Thereby we get that $x^h = x$ and $y^h = y$ or $x^h = y$ and $y^h = x$. For the first case we know $h = 1_H$ by Claim 1. For the second, we get that $x^{h^2} = x$. Therefore, $h^2 = 1_H$.

Now if there exists an element $g \in H_{(x,y)} \setminus \{1_H, h\}$, then we get $x^g = y = x^h$ and $y^g = x = y^h$. Thereby we get $g = h$ by Claim 1, a contradiction. So we get that $|H_{(x,y)}| = 2$.

Claim 4. *For any* $(x,y) \in E(G)$, *if* $|H_{(x,y)}| = 1$, *then* $(x,y)^H$ *is a 2-factor.*

Because $x^H = V(G) \subset V(\left\langle (x,y)^H \right\rangle) \subset V(G)$, so $V(\left\langle (x,y)^H \right\rangle) = V(G)$. Therefore, $(x,y)^H$ is a spanning subgraph of G.

Since H acting on $V(G)$ is transitive, there exists an element $h \in H$ such that $x^h = y$. It is obvious that $o(h)$ is finite and $o(h) \neq 2$. Otherwise, we have $|H_{(x,y)}| \geq 2$, a contradiction. Now $(x,y)^{\langle h \rangle} = x x^h x^{h^2} \cdots x^{h^{o(h)-1}} x$ is a circuit in the graph G. Consider the right coset decomposition of H on $\langle h \rangle$. Suppose $H = \bigcup\limits_{i=1}^{s} \langle h \rangle a_i$, $\langle h \rangle a_i \bigcap \langle h \rangle a_j = \emptyset$, if $i \neq j$, and $a_1 = 1_H$.

Now let $X = \{a_1, a_2, ..., a_s\}$. We know that for any $a, b \in X, (\langle h \rangle a) \bigcap (\langle h \rangle b) = \emptyset$ if $a \neq b$. Since $(x,y)^{\langle h \rangle a} = ((x,y)^{\langle h \rangle})^a$ and $(x,y)^{\langle h \rangle b} = ((x,y)^{\langle h \rangle})^b$ are also circuits, if $V(\left\langle (x,y)^{\langle h \rangle a} \right\rangle) \bigcap V(\left\langle (x,y)^{\langle h \rangle b} \right\rangle) \neq \emptyset$ for some $a, b \in X, a \neq b$, then there must be two elements $f, g \in \langle h\rangle$ such that $x^{fa} = x^{gb}$. According to Claim 1, we get that $fa = gb$, that is $ab^{-1} \in \langle h\rangle$. So $\langle h\rangle a = \langle h\rangle b$ and $a = b$, contradicts to the assumption that $a \neq b$.

Thereafter we know that $(x, y)^H = \bigcup\limits_{a\in X}(x, y)^{\langle h\rangle a}$ is a disjoint union of circuits. So $(x, y)^H$ is a 2-factor of the graph G.

Claim 5. *For any* $(x, y) \in E(G)$, $(x, y)^H$ *is an* 1*-factor if* $|H_{(x,y)}| = 2$.

Similar to the proof of Claim 4, we know that $V(\left\langle(x, y)^H\right\rangle) = V(G)$ and $(x, y)^H$ is a spanning subgraph of the graph G.

Let $H_{(x,y)} = \{1_H, h\}$, where $x^h = y$ and $y^h = x$. Notice that $(x, y)^a = (x, y)$ for $\forall a \in H_{(x,y)}$. Consider the coset decomposition of H on $H_{(x,y)}$, we know that $H = \bigcup\limits_{i=1}^{t} H_{(x,y)}b_i$, where $H_{(x,y)}b_i \bigcap H_{(x,y)}b_j = \emptyset$ if $i \neq j, 1 \leq i, j \leq t$. Now let $L = \{H_{(x,y)}b_i, 1 \leq i \leq t\}$. We get a decomposition

$$(x, y)^H = \bigcup_{b\in L}(x, y)^b$$

for $(x, y)^H$. Notice that if $b = H_{(x,y)}b_i \in L$, $(x, y)^b$ is an edge of G. Now if there exist two elements $c, d \in L, c = H_{(x,y)}f$ and $d = H_{(x,y)}g$, $f \neq g$ such that $V(\langle(x, y)^c\rangle) \bigcap V(\left\langle(x, y)^d\right\rangle) \neq \emptyset$, there must be $x^f = x^g$ or $x^f = y^g$. If $x^f = x^g$, we get $f = g$ by Claim 1, contradicts to the assumption that $f \neq g$. If $x^f = y^g = x^{hg}$, where $h \in H_{(x,y)}$, we get $f = hg$ and $fg^{-1} \in H_{(x,y)}$, so $H_{(x,y)}f = H_{(x,y)}g$. According to the definition of L, we get $f = g$, also contradicts to the assumption that $f \neq g$. Therefore, $(x, y)^H$ is an 1-factor of the graph G.

Now we can prove the assertion in this theorem. According to Claim 1- Claim 4, we get that

$$G = \left(\bigoplus_{y\in N_G(x),|H_{(x,y)}|=1}(x, y)^H\right)\bigoplus\left(\bigoplus_{y\in N_G(x),|H_{(x,y)}|=2}(x, y)^H\right).$$

for any chosen vertex $x, x \in V(G)$. By Claims 5 and 6, we know that $(x, y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and is a 1-factor if $|H_{(x,y)}| = 2$. Whence, the desired factorization for G is obtained. □

For a Cayley graph Cay($\Gamma : S$), by Theorem 2.2.2 we can always choose the vertex $x = 1_\Gamma$ and H the right regular transformation group on Γ. After then, Theorem 2.4.3 can be restated as follows.

Theorem 2.4.4 *Let* Γ *be a finite group with a subset* $S, S^{-1} = S$, $1_\Gamma \notin S$ *and* H *is the right transformation group on* Γ. *Then there is a factorization*

$$G = (\bigoplus_{s\in S, s^2\neq 1_\Gamma} (1_\Gamma, s)^H)\bigoplus(\bigoplus_{s\in S, s^2=1_\Gamma} (1_\Gamma, s)^H)$$

for the Cayley graph Cay$(\Gamma : S)$ *such that* $(1_\Gamma, s)^H$ *is a* 2*-factor if* $s^2 \neq 1_\Gamma$ *and* 1*-factor if* $s^2 = 1_\Gamma$.

Proof For any $h \in H_{(1_\Gamma,s)}$, if $h \neq 1_\Gamma$, then we get that $1_\Gamma h = s$ and $sh = 1_\Gamma$, that is $s^2 = 1_\Gamma$. According to Theorem 2.4.3, we get the factorization for the Cayley graph Cay$(\Gamma : S)$. □

§2.5 SMARANDACHE SEQUENCES ON SYMMETRIC GRAPHS

2.5.1 Smarandache Sequence with Symmetry. Let $\mathbb{Z}^+$ be the set of non-negative integers and Γ a group. We consider sequences $\{i(n)|n \in \mathbb{Z}^+\}$ and $\{g_n \in \Gamma|n \in \mathbb{Z}^+\}$ in this paper. There are many interesting sequences appeared in literature. For example, the sequences presented by Prof.Smarandache in references [Del1] and [Sma6] following:

(1) **Consecutive sequence**

1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, ⋯;

(2) **Digital sequence**

1, 11, 111, 1111, 11111, 11111, 1111111, 11111111, ⋯

(3) **Circular sequence**

1, 12, 21, 123, 231, 312, 1234, 2341, 3412, 4123, ⋯;

(4) **Symmetric sequence**

1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321, ⋯;

(5) **Divisor product sequence**

1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19, ⋯;

(6) **Cube-free sieve**

2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, ⋯.

Smarandache found the following nice symmetries for these integer sequences.

$$
\begin{aligned}
&1 \times 1 = 1\\
&11 \times 11 = 121\\
&111 \times 111 = 12321\\
&1111 \times 1111 = 1234321\\
&11111 \times 11111 = 12345431\\
&111111 \times 111111 = 12345654321\\
&1111111 \times 1111111 = 1234567654321\\
&11111111 \times 11111111 = 13456787654321\\
&111111111 \times 111111111 = 12345678987654321
\end{aligned}
$$

2.5.2 Smarandache Sequence on Symmetric Graph. Let $l_G^S : V(G) \to \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$ be a vertex labeling of a graph G with edge labeling $l_G^S(u, v)$ induced by $l_G^S(u)l_G^S(v)$ for $(u, v) \in E(G)$ such that $l_G^S(E(G)) = \{1, 121, 12321, 1234321, 123454321, 12345654321, 1234567654321, 123456787654321, 12345678987654321\}$, i.e., $l_G^S(V(G) \cup E(G))$ contains all numbers appeared in the Smarandache's symmetry. Denote all graphs with l_G^S labeling by $\mathscr{L}^S$. Then it is easily find a graph with a labeling l_G^S in Fig.2.5.1 following.

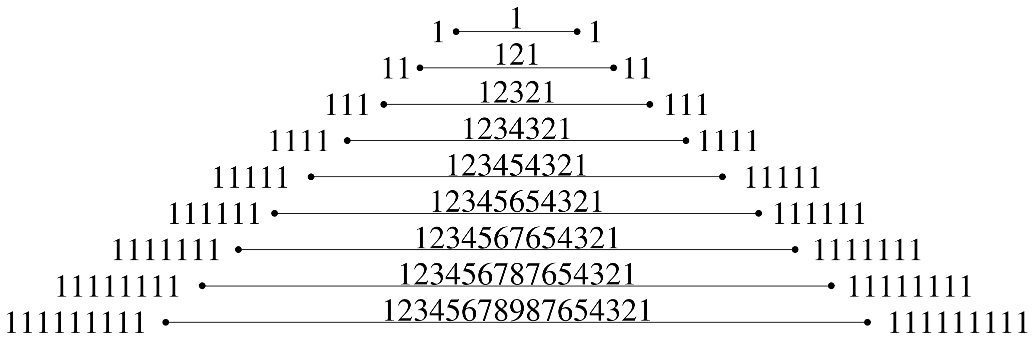


Fig.2.5.1

We know the following result.

Theorem 2.5.1 *Let $G \in \mathscr{L}^S$. Then $G = \bigcup\limits_{i=1}^{n} H_i$ for an integer $n \geq 9$, where each H_i is a connected graph. Furthermore, if G is vertex-transitive graph, then $G = nH$ for an integer $n \geq 9$, where H is a vertex-transitive graph.*

Proof Let $C(i)$ be the connected component with a label i for a vertex u, where

$i \in \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$. Then all vertices v in $C(i)$ must be with label $l_G^S(v) = i$. Otherwise, if there is a vertex v with $l_G^S(v) = j \in \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\} \setminus \{i\}$, let $P(u, v)$ be a path connecting vertices u and v. Then there must be an edge (x, y) on $P(u, v)$ such that $l_G^S(x) = i$, $l_G^S(y) = j$. By definition, $i \times j \notin l_G^S(E(G))$, a contradiction. So there are at least 9 components in G.

Now if G is vertex-transitive, we are easily know that each connected component $C(i)$ must be vertex-transitive and all components are isomorphic. $\square$

The smallest graph in $\mathscr{L}_v^S$ is the graph $9K_2$ shown in Fig.2.5.1. It should be noted that each graph in $\mathscr{L}_v^S$ is not connected. For finding a connected one, we construct a graph $\widetilde{Q}_k$ following on the digital sequence

$$1, 11, 111, 1111, 11111, \cdots, \underbrace{11\cdots1}_{k}.$$

by

$$V(\widetilde{Q}_k) = \{1, 11, \cdots, \underbrace{11\cdots1}_{k}\} \bigcup \{1', 11', \cdots, \underbrace{11\cdots1'}_{k}\},$$

$$E(\widetilde{Q}_k) = \{(1, \underbrace{11\cdots1}_{k}), (x, x'), (x, y) | x, y \in V(\widetilde{Q}) \text{ differ in precisely one } 1\}.$$

Now label $x \in V(\widetilde{Q})$ by $l_G(x) = l_G(x') = x$ and $(u, v) \in E(\widetilde{Q})$ by $l_G(u)l_G(v)$. Then we have the following result for the graph $\widetilde{Q}_k$.

Theorem 2.5.2 *For any integer* $m \geq 3$*, the graph* $\widetilde{Q}_m$ *is a connected vertex-transitive graph of order* $2m$ *with edge labels*

$$l_G(E(\widetilde{Q})) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence.

Proof Clearly, $\widetilde{Q}_m$ is connected. We prove it is a vertex-transitive graph. For simplicity, denote $\underbrace{11\cdots1}_{i}$, $\underbrace{11\cdots1'}_{i}$ by $\overline{i}$ and $\overline{i}'$, respectively. Then $V(\widetilde{Q}_m) = \{\overline{1}, \overline{2}, \cdots, \overline{m}\}$. We define an operation + on $V(\widetilde{Q}_k)$ by

$$\overline{k} + \overline{l} = \underbrace{11\cdots1}_{k+l(\mathrm{mod}k)} \quad \text{and} \quad \overline{k}' + \overline{l}' = \overline{k+l}', \quad \overline{k}'' = \overline{k}$$

for integers $1 \leq k, l \leq m$. Then an element $\overline{i}$ naturally induces a mapping

$$i^* : \ \overline{x} \rightarrow \overline{x+i}, \ \text{ for } \overline{x} \in V(\widetilde{Q}_m).$$

It should be noted that i^* is an automorphism of $\widetilde{Q}_m$ because tuples $\overline{x}$ and $\overline{y}$ differ in precisely one 1 if and only if $\overline{x+i}$ and $\overline{y+i}$ differ in precisely one 1 by definition. On the other hand, the mapping $\tau : \overline{x} \to \overline{x}'$ for $\forall \overline{x} \in$ is clearly an automorphism of $\widetilde{Q}_m$. Whence,

$$\mathscr{G} = \langle\ \tau, i^* \mid 1 \le i \le m\rangle \le \mathrm{Aut}\widetilde{\mathrm{Q}}_{\mathrm{m}},$$

which acts transitively on $V(\widetilde{Q})$ because $(\overline{y-x})^*(\overline{x}) = \overline{y}$ for $\overline{x}, \overline{y} \in V(\widetilde{Q}_m)$ and $\tau : \overline{x} \to \overline{x}'$.

Calculation shows easily that

$$l_G(E(\widetilde{Q}_m)) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence. This completes the proof. $\square$

By the definition of graph $\widetilde{Q}_m$, w consequently get the following result by Theorem 2.5.2.

Corollary 2.5.1 *For any integer $m \ge 3$, $\widetilde{Q}_m \simeq C_m \times P_2$.*

The smallest graph containing the third symmetry is $\widetilde{Q}_9$ shown in Fig.2.5.2 following,

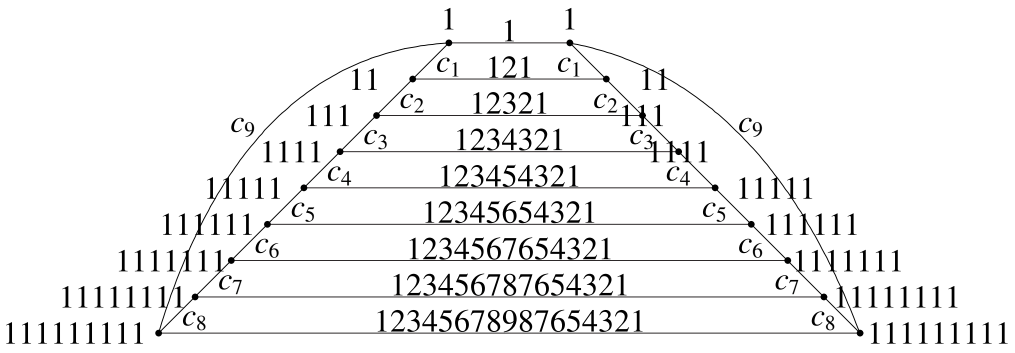


Fig.2.5.2

where $c_1 = 11$, $c_2 = 1221$, $c_3 = 123321$, $c_4 = 12344321$, $c_5 = 12344321$, $c_5 = 1234554321$, $c_6 = 123456654321$, $c_7 = 12345677654321$, $c_8 = 1234567887654321$, $c_9 = 123456789987654321$.

2.5.3 Group on Symmetric Graph. In fact, the Smarandache digital or symmetric sequences are subsequences of $\mathbb{Z}$, a special infinite Abelian group. We consider generalized labelings on vertex-transitive graphs following.

Problem 2.5.1 *Let* $(\Gamma;\circ)$ *be an Abelian group generated by* $x_1,\cdots,x_n$. *Thus* $\Gamma=\langle x_1,x_2,\cdots,x_n|W_1,\cdots\rangle$. *Find connected vertex-transitive graphs G with a labeling* $l_G: V(G)\to\{1_\Gamma,x_1,x_2,\cdots,x_n\}$ *and induced edge labeling* $l_G(u,v)=l_G(u)\circ l_G(v)$ *for* $(u,v)\in E(G)$ *such that*

$$l_G(E(G))=\{1_\Gamma,x_1^2,x_1\circ x_2,x_2^2,x_2\circ x_3,\cdots,x_{n-1}\circ x_n,x_n^2\}.$$

Similar to that of Theorem 2.5.2, we know the following result.

Theorem 2.5.3 *Let* $(\Gamma;\circ)$ *be an Abelian group generated by* $x_1,x_2,\cdots,x_n$ *for an integer* $n\geq 1$. *Then there are vertex-transitive graphs G with a labeling* $l_G: V(G)\to\{1_\Gamma,x_1,x_2,\cdots,x_n\}$ *such that the induced edge labeling by* $l_G(u,v)=l_G(u)\circ l_G(v)$ *with*

$$l_G(E(G))=\{1_\Gamma,x_1^2,x_1\circ x_2,x_2^2,x_2\circ x_3,\cdots,x_{n-1}\circ x_n,x_n^2\}.$$

Proof For any integer $m\geq 1$, define a graph $\widehat{Q}_{m,n,k}$ by

$$V(\widehat{Q}_{m,n,k})=\left(\bigcup_{i=0}^{m-1}U^{(i)}[x]\right)\bigcup\left(\bigcup_{i=0}^{m-1}W^{(i)}[y]\right)\bigcup\cdots\bigcup\left(\bigcup_{i=0}^{m-1}U^{(i)}[z]\right)$$

where $|\{U^{(i)}[x],v^{(i)}[y],\cdots,W^{(i)}[z]\}|=k$ and

$$\begin{aligned}
&U^{(i)}[x]=\{x_0^{(i)},x_1^{(i)},x_2^{(i)},\cdots,x_n^{(i)}\},\\
&V^{(i)}[y]=\{(y_0)^{(i)},y_1^{(i)},y_2^{(i)},\cdots,y_n^{(i)}\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&W^{(i)}[z]=\{(z_0)^{(i)},z_1^{(i)},z_2^{(i)},\cdots,z_n^{(i)}\}
\end{aligned}$$

for integers $0\leq i\leq m-1$, and

$$E(\widehat{Q}_{m,n})=E_1\bigcup E_2\bigcup E_3,$$

where $E_1=\{\,(x_l^{(i)},y_l^{(i)}),\cdots,(z_l^{(i)},x_l^{(i)})\mid 0\leq l\leq n-1,\ 0\leq i\leq m-1\}$, $E_2=\{\,(x_l^{(i)},x_{l+1}^{(i)}),(y_l^{(i)},y_{l+1}^{(i)}),\cdots,(z_l^{(i)},z_{l+1}^{(i)})\mid 0\leq l\leq n-1,\ 0\leq i\leq m-1,$ where $l+1\equiv(\mathrm{mod}n)\}$ and $E_3=\{(x_l^{(i)},x_l^{(i+1)}),(y_l^{(i)},y_l^{(i+1)}),\cdots,(z_l^{(i)},z_l^{(i+1)})|0\leq l\leq n-1,\ 0\leq i\leq m-1,$ where $i+1\equiv(\mathrm{mod}m)\}$. Then is clear that $\widehat{Q}_{m,n,k}$ is connected. We prove this graph is vertex-transitive. In fact, by defining three mappings

$$\begin{aligned}
&\theta: x_l^{(i)}\to x_{l+1}^{(i)},\ y_l^{(i)}\to y_{l+1}^{(i)},\cdots,z_l^{(i)}\to z_{l+1}^{(i)},\\
&\tau: x_l^{(i)}\to y_l^{(i)},\cdots,z_l^{(i)}\to x_l^{(i)},
\end{aligned}$$

$$\sigma : x_l^{(i)} \to x_l^{(i+1)},\ y_l^{(i)} \to y_l^{(i+1)}, \cdots, z_l^{(i)} \to z_l^{(i+1)},$$

where $1 \leq l \leq n,\ 1 \leq i \leq m,\ i+1(\mathrm{mod} m),\ l+1(\mathrm{mod} n)$. Then it is easily to check that $\theta,\ \tau$ and σ are automorphisms of the graph $\widehat{Q}_{m,n,k}$ and the subgroup $\langle\theta, \tau, \sigma\rangle$ acts transitively on $V(\widehat{Q}_{m,n,k})$.

Now we define a labeling $l_{\widehat{Q}}$ on vertices of $\widehat{Q}_{m,n,k}$ by

$$l_{\widehat{Q}}(x_0^{(i)}) = l_{\widehat{Q}}(y_0^{(i)}) = \cdots = l_{\widehat{Q}}(z_0^{(i)}) = 1_\Gamma,$$
$$l_{\widehat{Q}}(x_l^{(i)}) = l_{\widehat{Q}}(y_l^{(i)}) = \cdots = l_{\widehat{Q}}(z_l^{(i)}) = x_l, \quad 1 \leq i \leq m,\ 1 \leq l \leq n.$$

Then we know that $l_G(E(G)) = \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ and

$$l_G(E(G)) = \{1_\Gamma, x_1^2, x_1 \circ x_2, x_2^2, x_2 \circ x_3, \cdots, x_{n-1} \circ x_n, x_n^2\}. \qquad \square$$

Particularly, let Γ be a subgroup of $(\mathbb{Z}_{111111111}, \times)$ generated by

$$\{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$$

and $m = 1$. We get the symmetric sequence on a symmetric graph shown in Fig.2.5.2 again. Let $m = 5, n = 3$ and $k = 2$, i.e., the graph $\widehat{Q}_{5,3,2}$ with a labeling $l_G : V(\widehat{Q}_{5,3,2}) \to \{1_\Gamma, x_1, x_2, x_3, x_4\}$ is shown in Fig.2.5.3 following.

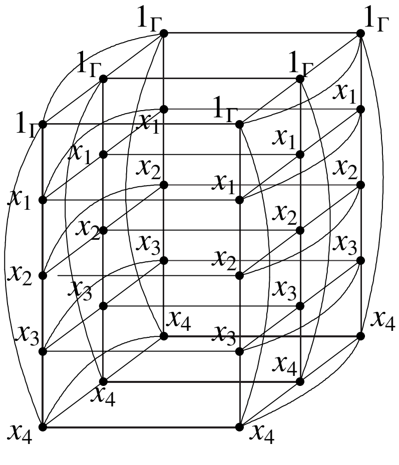


Fig.4.1

Denote by $N_G[x]$ all vertices in a graph G labeled by an element $x \in \Gamma$. Then we immediately get results following by the proof of Theorem 2.5.3.

Corollary 2.5.2 *For integers* $m, n \geq 1$, $\widehat{Q}_{m,n,k} \simeq C_m \times C_n \times C_k$.

Corollary 2.5.3 $|N_{\widehat{Q}_{m,n,k}}[x]| = mk$ *for* $\forall x \in \{1_\Gamma, x_1, \cdots, x_n\}$ *and integers* $m, n, k \geq 1$.

§2.6 RESEARCH PROBLEMS

2.6.1 For catering to the need of computer science, graphs were out of games and turned into a theory for dealing with objects with relations in last century. There are many excellent monographs for its theoretical results with applications, such as these references [BoM1], [ChL1], [GoR1] and [Whi1] for graphs with structures and [GrT1], [MoT1] and [Liu1]-[Liu3] for graphs on surfaces.

2.6.2 A graph property P is *Smarandachely* if it behaves in at least two different ways on a graph, i.e., validated and invalided, or only invalided but in multiple distinct ways. Such a graph with at least one Smarandachely graph property is called a *Smarandachely graph*. Whence, one can generalizes conceptions in graphs by this Smarandache notion. We list such conceptions with open problems following.

Smarandachely k-Constrained Labeling. A *Smarandachely k-constrained labeling* of a graph $G(V, E)$ is a bijective mapping $f : V \cup E \to \{1, 2, .., |V| + |E|\}$ with the additional conditions that $|f(u) - f(v)| \geq k$ whenever $uv \in E$, $|f(u) - f(uv)| \geq k$ and $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a Smarandachely k-constrained total graph, abbreviated as $k - CTG$. An example for $k = 5$ on P_7 is shown in Fig.2.6.1.

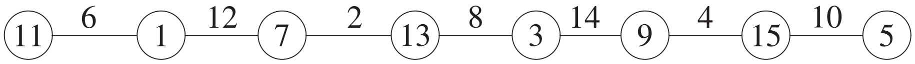


Fig.2.6.1

The minimum positive integer n such that the graph $G \cup \overline{K_n}$ is a $k - CTG$ is called *k-constrained number* of the graph G and denoted by $t_k(G)$.

Problem 2.6.1 *Determine $t_k(G)$ for a graph G.*

Smarandachely Super m-Mean Labeling. Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced *Smarandachely edge m-labeling* f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in$

$E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph. Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

A Smarandache super 2-mean labeling on P_6^2 is shown in Fig.2.6.2.

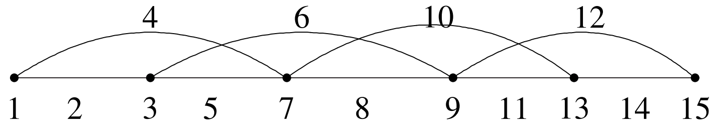


Fig.2.6.2

Problem 2.6.2 *Determine which graph G possesses a Smarandachely super m-mean labeling.*

Smarandachely Λ-Coloring. Let Λ be a subgraph of a graph G. A *Smarandachely* Λ*-coloring* of a graph G by colors in $\mathscr{C}$ is a mapping $\varphi_\Lambda : \mathscr{C} \to V(G) \cup E(G)$ such that $\varphi(u) \neq \varphi(v)$ if u and v are elements of a subgraph isomorphic to Λ in G. Similarly, a Smarandachely Λ-coloring $\varphi_\Lambda|_{V(G)} : \mathscr{C} \to V(G)$ or $\varphi_\Lambda|_{E(G)} : \mathscr{C} \to E(G)$ is called a *vertex Smarandachely* Λ*-coloring* or an *edge Smarandachely* Λ*-coloring*.

Problem 2.6.3 *For a graph G and* $\Lambda \prec G$*, determine the numbers* $\chi^\Lambda(G)$ *and* $\chi_1^\Lambda(G)$.

Smarandachely $(\mathscr{P}_1, \mathscr{P}_2)$**-Decomposition.** Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be graphical properties. A *Smarandachely* $(\mathscr{P}_1, \mathscr{P}_2)$*-decomposition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \cdots, G_l \in \mathscr{P}$ such that $G_i \in \mathscr{P}_1$ or $G_i \notin \mathscr{P}_2$ for integers $1 \leq i \leq l$. If $\mathscr{P}_1$ or $\mathscr{P}_2 = \{\text{all graphs}\}$, a Smarandachely $(\mathscr{P}_1, \mathscr{P}_2)$-decomposition of a graph G is said to be a *Smarandachely* $\mathscr{P}$*-decomposition*. The minimum cardinality of Smarandachely $(\mathscr{P}_1, \mathscr{P}_2)$-decomposition are denoted by $\Pi_{\mathscr{P}_1,\mathscr{P}_2}(G)$.

Problem 2.6.4 *For a graph G and properties* $\mathscr{P}$*,* $\mathscr{P}'$*, determine* $\Pi_{\mathscr{P}}(G)$ *and* $\Pi_{\mathscr{P},\mathscr{P}'}(G)$.

2.6.3 Smarandache also found the following two symmetries on digits:

$$\begin{array}{ll}
1\times 8+1=9 & 1\times 9+2=11\\
12\times 8+2=98 & 12\times 9+3=111\\
123\times 8+3=987 & 123\times 9+4=1111\\
1234\times 8+4=9876 & 1234\times 9+5=11111\\
12345\times 8+5=98765 & 12345\times 9+6=111111\\
123456\times 8+6=987654 & 123456\times 9+7=1111111\\
1234567\times 8+7=9876543 & 1234567\times 9+8=11111111\\
12345678\times 8+8=98765432 & 12345678\times 9+9=111111111\\
123456789\times 8+9=987654321 & 123456789\times 9+10=1111111111
\end{array}$$

Thus we can also label vertices $l_{V(G)}: V(G)\rightarrow\mathscr{C}$ of a graph by consecutive sequence $\mathscr{C}$ with an induced edge labeling $l_{E(G)}(uv)=cl_{V(G)}(u)+l_{G(V)}$ for $\forall uv\in E(G)$, where c is a chosen digit. For example, let $l_{V(G)}=\{1,2,3,4,5,6,7,8,9,12,123,1234,12345,123456,1234567,12345678,123456789\}$, $c=8$ or $l_{V(G)}=\{1,2,3,4,5,6,7,8,9,10,12,123,1234,12345,123456,1234567,12345678,123456789\}$, $c=9$, we can extend these previous digital symmetries on symmetric graphs with digits. Generally, there is an open problem following.

Problem 2.6.5 *Let $(\mathscr{A};+,\cdot)$ be an algebraic system with operations $+$, $\cdot$. Find graphs G with vertex labeling $l_{V(G)}: V(G)\rightarrow\mathscr{A}$ and edge labeling $l_{E(G)}(uv)=c_1\cdot l_{V(G)}(u)+c_2\cdot l_{V(G)}(v)$ (or $l_{E(G)}(uv)=(c_1+l_{V(G)}(u))\cdot(c_2+l_{V(G)}(v))$) for $c_1,\ c_2\in\mathscr{A}$, $\forall uv\in E(G)$ such that they are both symmetric in graph and element.*

*Particularly, let $\mathscr{T}$ be a set of symmetric elements in $\mathscr{A}$. For example, $\mathscr{T}=\{\ a\cdot b, b\cdot a\ |\ a,b\in\mathscr{A}\ \}$. Find symmetric graphs with vertex labeling $l_{V(G)}: V(G)\rightarrow\mathscr{T}$ and edge labeling $l_{E(G)}(uv)=l_{V(G)}(u)+l_{V(G)}(v)$ (or $l_{E(G)}(uv)=l_{V(G)}(u)\cdot l_{V(G)}(v)$) such that $l_{V(G)}(u)+l_{V(G)}(v)$ (or $l_{E(G)}(uv)=l_{V(G)}(u)\cdot l_{V(G)}(v)$) is itself a symmetric element in $\mathscr{A}$ for $\forall uv\in E(G)$, for example, the labeled graph shown in Fig.*2.5.2.

CHAPTER 3.

Algebraic Multi-Spaces

Accompanied with humanity into the 21st century, a highlight trend for developing a science is its overlap and hybrid, and harmoniously with other sciences. Algebraic systems, such as those of operation systems, groups, rings, fields, vector spaces and modules characterize algebraic structures on sets, which are discrete representations for phenomena in the natural world. The notion of multi-space enables one to construct algebraic multi-structures and discusses multi-systems, multi-groups, multi-rings, multi-fields, vector multi-spaces and multi-modules in this chapter, maybe completed or not in cases. These algebraic multi-spaces also show us that a theorem in mathematics is truth under conditions, i.e., a lateral feature on mathematical systems. Certainly, more consideration should be done on these algebraic multi-spaces, especially, by an analogous thinking as those in classical algebra. For this objective, a few open problems on algebraic multi-spaces ca be found in final section of this chapter.

§3.1 ALGEBRAIC MULTI-STRUCTURES

3.1.1 Algebraic Multi-Structure. Algebraic systems, such as those of group, ring, field, linear algebra, etc. enable one to construct algebraic multi-structures and raise the following definition by Smarandache's notion.

Definition 3.1.1 *An algebraic multi-system is a pair* $\left(\widetilde{A};\widetilde{O}\right)$ *with a set* $\widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$ *and an operation set*

$$\widetilde{O} = \{\circ_i \mid 1 \leq i \leq n\}$$

on $\widetilde{A}$ *such that each pair* $(A_i;\circ_i)$ *is an algebraic system.*

A multi-system $\left(\widetilde{A};\widetilde{O}\right)$ is *associative* if for $\forall a, b, c \in \widetilde{A}$, $\forall \circ_1, \circ_2 \in \widetilde{O}$, there is

$$(a \circ_1 b) \circ_2 c = a \circ_1 (b \circ_2 c).$$

Such a system is called an *associative multi-system*.

Let $\left(\widetilde{A};\widetilde{O}\right)$ be a multi-system and $\widetilde{B} \subset \widetilde{A}$, $\widetilde{Q} \subset \widetilde{O}$. If $\left(\widetilde{B};\widetilde{Q}\right)$ is itself a multi-system, we call $\left(\widetilde{B};\widetilde{Q}\right)$ a *multi-subsystem of* $\left(\widetilde{A};\widetilde{O}\right)$, denoted by $\left(\widetilde{B};\widetilde{Q}\right) \prec \left(\widetilde{A};\widetilde{O}\right)$.

Assume $\left(\widetilde{B};\widetilde{O}\right) \prec \left(\widetilde{A};\widetilde{O}\right)$. For $\forall a \in \widetilde{A}$ and $\circ_i \in \widetilde{O}$, where $1 \leq i \leq l$, define a coset $a \circ_i \widetilde{B}$ by

$$a \circ_i \widetilde{B} = \{\, a \circ_i b \mid \text{for } \forall b \in \widetilde{B}\},$$

and let

$$\widetilde{A} = \bigcup_{a \in R, \circ \in \widetilde{P} \subset \widetilde{O}} a \circ \widetilde{B}.$$

Then the set

$$\mathscr{Q} = \left\{\, a \circ \widetilde{B} \mid a \in R, \circ \in \widetilde{P} \subset \widetilde{O}\right\}$$

is called a *quotient set of* $\widetilde{B}$ *in* $\widetilde{A}$ *with a representation pair* $(R, \widetilde{P})$, denoted by $\widetilde{A}/\widetilde{B}\big|_{(R,\widetilde{P})}$.

Two multi-systems $\left(\widetilde{A}_1;\widetilde{O}_1\right)$ and $\left(\widetilde{A}_2;\widetilde{O}_2\right)$ are called *homomorphic* if there is a mapping $\omega : \widetilde{A}_1 \to \widetilde{A}_2$ with $\omega : \widetilde{O}_1 \to \widetilde{O}_2$ such that for $a_1, b_1 \in \widetilde{A}_1$ and $\circ_1 \in \widetilde{O}_1$, there exists an operation $\circ_2 = \omega(\circ_1) \in \widetilde{O}_2$ enables that

$$\omega(a_1 \circ_1 b_1) = \omega(a_1) \circ_2 \omega(b_1).$$

Similarly, if ω is a bijection, $\left(\widetilde{A}_1;\widetilde{O}_1\right)$ and $\left(\widetilde{A}_2;\widetilde{O}_2\right)$ are called *isomorphic*, and if $\widetilde{A}_1 = \widetilde{A}_2 = \widetilde{A}$, ω is called an *automorphism on* $\widetilde{A}$.

For a binary operation "$\circ$", if there exists an element $1_\circ^l$ (or $1_\circ^r$) such that

$$1_\circ^l \circ a = a \text{ or } a \circ 1_\circ^r = a$$

for $\forall a \in A_i, 1 \leq i \leq n$, then $1_\circ^l$ ($1_\circ^r$) is called a *left (right) unit*. If $1_\circ^l$ and $1_\circ^r$ exist simultaneously, then there must be

$$1_\circ^l = 1_\circ^l \circ 1_\circ^r = 1_\circ^r = 1_\circ.$$

Call $1_\circ$ a *unit* of A_i.

Remark 3.1.1 In Definition 3.1.1, the following three cases are permitted:

(1) $A_1 = A_2 = \cdots = A_n$, i.e., n operations on one set.

(2) $\circ_1 = \circ_2 = \cdots = \circ_n$, i.e., n set with one law.

3.1.2 Example. Some examples for multi-system are present in the following.

Example 3.1.1 Take n disjoint two by two cyclic groups $C_1, C_2, \cdots, C_n$, $n \geq 2$ with

$$C_1 = (\langle a \rangle\, ; +_1), C_2 = (\langle b \rangle\, ; +_2), \cdots, C_n = (\langle c \rangle\, ; +_n),$$

where "$+_1, +_2, \cdots, +_n$" are n binary operations. Then their union $\widetilde{C} = \bigcup\limits_{i=1}^{n} C_i$ is a multi-space. In this multi-space, for $\forall x, y \in \widetilde{C}$, if $x, y \in C_k$ for some integer k, then we know $x +_k y \in C_k$. But if $x \in C_s$, $y \in C_t$ and $s \neq t$, then we do not know which binary operation between them and what is the resulting element corresponding to them.

Example 3.1.2 Let $(G; \circ)$ be a group with a binary operation "$\circ$". Choose n different elements $h_1, h_2, \cdots, h_n$, $n \geq 2$ and make the extension of the group $(G; \circ)$ by $h_1, h_2, \cdots, h_n$ respectively as follows:

$(G \bigcup \{h_1\}; \times_1)$, where the binary operation $\times_1 = \circ$ for elements in G, otherwise, new operation;

$(G \bigcup \{h_2\}; \times_2)$, where the binary operation $\times_2 = \circ$ for elements in G, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots$;

$(G \bigcup \{h_n\}; \times_n)$, where the binary operation $\times_n = \circ$ for elements in G, otherwise, new operation.

Define

$$\widetilde{G} = \bigcup_{i=1}^{n} \left(G \bigcup \{h_i\}; \times_i \right).$$

Then $\widetilde{G}$ is a multi-space with binary operations "$\times_1, \times_2, \cdots, \times_n$" . In this multi-space, for $\forall x, y \in \widetilde{G}$, we know the binary operation between x, y and the resulting element unless the exception cases $x = h_i, y = h_j$ with $i \neq j$.

For $n = 3$, such a multi-space is shown in Fig.3.1.1, in where the central circle represents the group G and each angle field the extension of G. Whence, this kind of multi-space is called a *fan multi-space*.

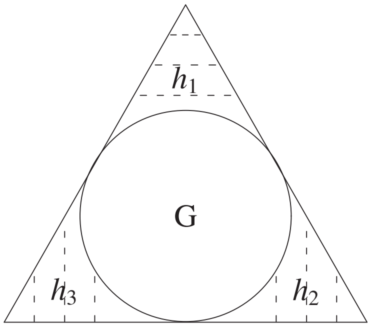


Fig.3.1.1

Similarly, we can also use a ring R to get fan multi-spaces. For example, let $(R; +, \circ)$ be a ring and let $r_1, r_2, \cdots, r_s$ be two by two different elements. Make these extensions of $(R; +, \circ)$ by $r_1, r_2, \cdots, r_s$ respectively as follows:

$(R \bigcup \{r_1\}; +_1, \times_1)$, where binary operations $+_1 = +$, $\times_1 = \circ$ for elements in R, otherwise, new operation;

$(R \bigcup \{r_2\}; +_2, \times_2)$, where binary operations $+_2 = +$, $\times_2 = \circ$ for elements in R, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots$;

$(R \bigcup \{r_s\}; +_s, \times_s)$, where binary operations $+_s = +$, $\times_s = \circ$ for elements in R, otherwise, new operation.

Define

$$\widetilde{R} = \bigcup_{j=1}^{s} \left(R \bigcup \{r_j\}; +_j, \times_j \right).$$

Then $\widetilde{R}$ is a fan multi-space with ring-like structure. Also we can define a fan multi-space with field-like, vector-like, semigroup-like,$\cdots$, etc. multi-structures.

These multi-spaces constructed in Examples 3.1.1 and 3.1.2 are not *completed*, i.e., there exist some elements in this space have not binary operation between them. In algebra, we wish to construct a *completed multi-space*, i.e., there is a binary operation between any two elements at least and their resulting is still in this space. The following examples constructed by applying *Latin squares* are such multi-spaces.

Example 3.1.3 Let S be a finite set with $|S| = n \geq 2$. Construct an $n \times n$ Latin square by elements in S, i.e., every element just appears one time on its each row and column. Choose k Latin squares $M_1, M_2, \cdots, M_k$, $k \leq \prod\limits_{s=1}^{n} s!$.

By a result in reference [Rys1], there are at least $\prod\limits_{s=1}^{n} s!$ distinct $n \times n$ Latin squares. Whence, we can always choose $M_1, M_2, \cdots, M_k$ distinct two by two. For a Latin square M_i, $1 \leq i \leq k$, define an operation "$\times_i$" by

$$\times_i : (s, f) \in S \times S \rightarrow (M_i)_{sf}.$$

For example, if $n = 3$, then $S = \{1, 2, 3\}$ and there are 2 Latin squares L_1, L_2 with

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Therefore, we get operations "$\times_1$" and "$\times_2$" in Table 3.1.1 by Latin squares L_1, L_2 following.

$\times_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$\times_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Table 1.3.1

Generally, for $\forall x, y, z \in S$ and two operations "$\times_i$" and "$\times_j$" , $1 \leq i, j \leq k$ define

$$x \times_i y \times_j z = (x \times_i y) \times_j z.$$

For example, if $n = 3$, then

$$1 \times_1 2 \times_2 3 = (1 \times_2) \times_2 3 = 2 \times_2 3 = 2$$

and

$$2 \times_1 3 \times_2 2 = (2 \times_1 3) \times_2 2 = 1 \times_2 3 = 3.$$

Thus S is a completed multi-space with k operations.

Notice that $\mathrm{Aut}Z_n \simeq Z_n^*$, where Z_n^* is the group of reduced residue class mod$n$ under the multiply operation. It is known that $|\mathrm{Aut}Z_n| = \varphi(n)$, where $\varphi(n)$ is the Euler function. Thus the automorphism group of the multi-space $\widetilde{C}$ in Example 3.1.1 is

$$\mathrm{Aut}\widetilde{C} = S_n[Z_n^*].$$

Whence, $|\mathrm{Aut}\widetilde{C}| = \varphi(n)^n n!$. For determining the automorphism groups of multi-spaces in Example 3.1.3 is an interesting problem for combinatorial design. The following example also constructs completed multi-spaces by algebraic systems.

Example 3.1.4 For constructing a completed multi-space, let $(S\ ;\circ)$ be an algebraic system, i.e., $a \circ b \in S$ for $\forall a, b \in S$. Whence, we can take $C, C \subseteq S$ being a cyclic group. Now consider a partition of S

$$S = \bigcup_{k=1}^{m} G_k$$

with $m \geq 2$ such that $G_i \bigcap G_j = C$ for $\forall i, j, 1 \leq i, j \leq m$.

For an integer $k, 1 \leq k \leq m$, assume $G_k = \{g_{k1}, g_{k2}, \cdots, g_{kl}\}$. Define an operation "$\times_k$" on G_k as follows, which enables $(G_k; \times_k)$ to be a cyclic group.

$$g_{k1} \times_k g_{k1} = g_{k2},$$

$$g_{k2} \times_k g_{k1} = g_{k3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$g_{k(l-1)} \times_k g_{k1} = g_{kl},$$

and

$$g_{kl)} \times_k g_{k1} = g_{k1}.$$

Then $S = \bigcup\limits_{k=1}^{m} G_k$ is a completed multi-space with $m + 1$ operations. The approach enables one to construct complete multi-spaces $\widetilde{A} = \bigcup\limits_{i=1}^{n}$ with k operations for $k \geq n + 1$.

3.1.3 Elementary Property. First, we introduce the following definition.

Definition 3.1.2 *A mapping f on a set X is called faithful if $f(x) = x$ for $\forall x \in X$, then $f = 1_X$, the unit mapping on X fixing each element in X.*

Notice that if f is faithful and $f_1(x) = f(x)$ for $\forall x \in X$, then $f_1^{-1}f = 1_X$, i.e., $f_1 = f$.

For each operation "$\times$" and a chosen element g in a subspace $A_i, A_i \subset \widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$, there is a *left-mapping* $f_g^l : A_i \to A_i$ defined by

$$f_g^l : a \to g \times a, \quad a \in A_i.$$

Similarly, we can define the *right-mapping* f_g^r.

Convention 3.1.1 *Each operation "$\times$" in a subset $A_i, A_i \subset \widetilde{A}$ with $\widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$ is faithful, i.e., for $\forall g \in A_i$, $\varsigma : g \to f_g^l$ (or $\tau : g \to f_g^r$) is faithful.*

Define the kernel $Ker\varsigma$ of a mapping ς by

$$\mathrm{Ker}\varsigma = \{g | g \in A_i \text{ and } \varsigma(g) = 1_{A_i}\}.$$

Then Convention 3.1.1 is equivalent to the following.

Convention 3.1.2 *For each $\varsigma : g \to f_g^l$ (or $\varsigma : g \to f_g^r$) induced by an operation "$\times$" has kernel*

$$\mathrm{Ker}\varsigma = \{1_\times^l\}$$

if $1_\times^l$ exists. Otherwise, $\mathrm{Ker}\varsigma = \emptyset$.

We get results following on multi-spaces $\widetilde{A}$.

Theorem 3.1.1 *For a multi-space $\left(\widetilde{A}; \widetilde{O}\right)$ with $\widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$ and an operation "$\times$", the left unit $1_\times^l$ and right unit $1_\times^r$ are unique if they exist.*

Proof If there are two left units $1_\times^l, I_\times^l$ in a subset A_i of a multi-space $\widetilde{A}$, then for $\forall x \in A_i$, their induced left-mappings $f_{1_\times^l}^l$ and $f_{I_\times^l}^l$ satisfy

$$f_{1_\times^l}^l(x) = 1_\times^l \times x = x, \quad f_{I_\times^l}^l(x) = I_\times^l \times x = x.$$

Therefore, we get that $f_{1_\times^l}^l = f_{I_\times^l}^l$. Since the mappings $\varsigma_1 : 1_\times^l \to f_{1_\times^l}^l$ and $\varsigma_2 : I_\times^l \to f_{I_\times^l}^l$ are faithful, we know that $1_\times^l = I_\times^l$. Similarly, we can also prove that the right unit $1_\times^r$ is also unique. □

For two elements a, b in multi-space $\widetilde{A}$, if $a \times b = 1_\times^l$, then b is called a *left-inverse* of a. If $a \times b = 1_\times^r$, then a is called a *right-inverse* of b. Certainly, if $a \times b = 1_\times$, then a is called an *inverse* of b and b an *inverse* of a.

Theorem 3.1.2 *For a multi-space* $\left(\widetilde{A}; \widetilde{O}\right)$ *with* $\widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$, $a \in \mathscr{H}$, *the left-inverse and right-inverse of a are unique if they exist.*

Proof Notice that $\kappa_a : x \to ax$ is faithful, i.e., Ker$\kappa = \{1_\times^l\}$ for $1_\times^l$ existing now.

If there exist two left-inverses b_1, b_2 in $\mathscr{H}$ such that $a \times b_1 = 1_\times^l$ and $a \times b_2 = 1_\times^l$, then we know that $b_1 = b_2 = 1_\times^l$. Similarly, we can also prove that the right-inverse of a is also unique. □

Corollary 3.1.1 *If "×" is an operation of a multi-space* $\mathscr{H}$ *with unit* $1_\times$, *then the equation* $a \times x = b$ *has at most one solution for the indeterminate* x.

Proof According to Theorem 3.1.2, there is at most one left-inverse a_1 of a such that $a_1 \times a = 1_\times$. Whence, we know that $x = a_1 \times a \times x = a_1 \times b$. □

We also get a consequence for solutions of an equation in a multi-space by this result.

Corollary 3.1.2 *Let* $\left(\widetilde{A}; \widetilde{O}\right)$ *be a multi-space. Then the equationa* $a \circ x = b$ *has at most* $|\widetilde{O}|$ *solutions, where* $\circ \in \widetilde{O}$.

§3.2 MULTI-GROUPS

3.2.1 Multi-Group. Let $\widetilde{G}$ be a set with binary operations $\widetilde{O}$. By definition $\left(\widetilde{G}; \widetilde{O}\right)$ is an *algebraic multi-system* if for $\forall a, b \in \widetilde{G}$ and $\circ \in \widetilde{O}$, $a \circ b \in \widetilde{G}$ provided $a \circ b$ existing.

Definition 3.2.1 *For an integer* $n \geq 1$, *an algebraic multi-system* $\left(\widetilde{G}; \widetilde{O}\right)$ *is an n-multi-group for an integer* $n \geq 1$ *if there are* $G_1, G_2, \cdots, G_n \subset \widetilde{G}$, $\widetilde{O} = \{\circ_i,\ 1 \leq i \leq n\}$ *with*

(1) $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$;
(2) $(G_i; \circ_i)$ *is a group for* $1 \leq i \leq n$.

For $\forall \circ \in \widetilde{O}$, denoted by $G_\circ$ the group $(G; \circ)$ and $G_\circ^{\max}$ the *maximal group* $(G; \circ)$, i.e., $(G_\circ^{\max}; \circ)$ is a group but $(G_\circ^{\max} \cup \{x\}; \circ)$ is not for $\forall x \in \widetilde{G} \setminus G_\circ^{\max}$ in $\left(\widetilde{G}; \widetilde{O}\right)$.

A distributed multi-group is such a multi-group with distributive laws hold for some operations, formally defined in the following.

Definition 3.2.2 *Let* $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ *be a complete multi-space with an operation set* $O\left(\widetilde{G}\right) = \{\times_i, 1 \leq i \leq n\}$. *If* $(G_i; \times_i)$ *is a group for any integer* $i, 1 \leq i \leq n$ *and for* $\forall x, y, z \in \widetilde{G}$ *and* $\forall \times, \circ \in O(\widetilde{G}), \times \neq \circ$, *there is one operation, for example the operation "*$\times$*" satisfying the distribution law to the operation "*$\circ$*" provided all of these operating results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$
$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ *is called a distributed multi-group.*

Remark 3.2.1 The following special cases for $n = 2$ convince us that distributed multi-groups are a generalization of groups, skew fields, fields, $\cdots$, etc..

(1) If $G_1 = G_2 = \widetilde{G}$ are groups, then $\widetilde{G}$ is a skew field.
(2) If $(G_1; \times_1)$ and $(G_2; \times_2)$ are commutative groups, then $\widetilde{G}$ is a field.

Definition 3.2.3 *Let* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ *and* $\left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$ *be multi-groups. Then* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ *is isomorphic to* $\left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$, *denoted by* $(\vartheta, \iota) : \left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right) \to \left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$ *if there are bijections* $\vartheta : \widetilde{\mathscr{G}_1} \to \widetilde{\mathscr{G}_2}$ *and* $\iota : \widetilde{O_1} \to \widetilde{O_2}$ *such that for* $a, b \in \widetilde{\mathscr{G}_1}$ *and* $\circ \in \widetilde{O_1}$, $\vartheta(a \circ b) = \vartheta(a)\iota(\circ)\vartheta(b)$ *provided* $a \circ b$ *existing in* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$. *Such isomorphic multi-groups are denoted by* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right) \simeq \left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$

Clearly, if $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ is an n-multi-group with (ϑ, ι) an isomorphism, the image of (ϑ, ι) is also an n-multi-group. Now let $(\vartheta, \iota) : \left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right) \to \left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$ with $\widetilde{\mathscr{G}_1} = \bigcup\limits_{i=1}^{n} \mathscr{G}_{1i}$, $\widetilde{\mathscr{G}_2} = \bigcup\limits_{i=1}^{n} \mathscr{G}_{2i}$, $\widetilde{O_1} = \{\circ_{1i},\ 1 \leq i \leq n\}$ and $\widetilde{O_2} = \{\circ_{2i},\ 1 \leq i \leq n\}$, then for $\circ \in \widetilde{O}$, $\mathscr{G}_{\circ}^{\max}$ is isomorphic to $\vartheta(\mathscr{G})_{\iota(\circ)}^{\max}$ by definition. The following result shows that its converse is also true.

Theorem 3.2.1 *Let* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ *and* $\left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$ *be n-multi-groups with*

$$\widetilde{\mathscr{G}_1} = \bigcup_{i=1}^{n} \mathscr{G}_{1i}, \quad \widetilde{\mathscr{G}_2} = \bigcup_{i=1}^{n} \mathscr{G}_{2i},$$

$\widetilde{O_1} = \{\circ_{i1},\ 1 \leq i \leq n\}$, $\widetilde{O_2} = \{\circ_{i2},\ 1 \leq i \leq n\}$. *If* $\phi_i : \mathscr{G}_{1i} \to \mathscr{G}_{2i}$ *is an isomorphism for each integer* $i,\ 1 \leq i \leq n$ *with* $\phi_k|_{\mathscr{G}_{1k} \cap \mathscr{G}_{1l}} = \phi_l|_{\mathscr{G}_{1k} \cap \mathscr{G}_{1l}}$ *for integers* $1 \leq k, l \leq n$, *then* $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ *is isomorphic to* $\left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$.

Proof Define mappings $\vartheta : \widetilde{\mathscr{G}_1} \to \widetilde{\mathscr{G}_2}$ and $\iota : \widetilde{O_1} \to \widetilde{O_1}$ by

$$\vartheta(a) = \phi_i(a) \text{ if } a \in \mathscr{G}_i \subset \widetilde{\mathscr{G}} \text{ and } \iota(\circ_{1i}) = \circ_{2i} \text{ for each integer } 1 \le i \le n.$$

Notice that $\phi_k|_{\mathscr{G}_{1k}\cap\mathscr{G}_{1l}} = \phi_l|_{\mathscr{G}_{1k}\cap\mathscr{G}_{1l}}$ for integers $1 \le k, l \le n$. We know that ϑ, ι both are bijections. Let $a, b \in \mathscr{G}_{1s}$ for an integer s, $1 \le s \le n$. Then

$$\vartheta(a \circ_{1s} b) = \phi_s(a \circ_{1s} b) = \phi_s(a) \circ_{2s} \phi_s(b) = \vartheta(a)\iota(\circ_{1s})\vartheta(b).$$

Whence, $(\vartheta, \iota) : \left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right) \to \left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$. □

3.2.2 Multi-Subgroup. Let $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ be a multi-group, $\widetilde{\mathscr{H}} \subset \widetilde{\mathscr{G}}$ and $O \subset \widetilde{O}$. If $\left(\widetilde{\mathscr{H}}; O\right)$ is multi-group itself, then $(\mathscr{H}; O)$ is called a multi-subgroup, denoted by $\left(\widetilde{\mathscr{H}}; O\right) \le \left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$. Then the following criterion is clear for multi-subgroups.

Theorem 3.2.2 *An multi-subsystem* $\left(\widetilde{\mathscr{H}}; O\right)$ *of a multi-group* $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ *is a multi-subgroup if and only if* $\widetilde{\mathscr{H}} \cap \mathscr{G}_\circ \le \mathscr{G}_\circ^{\max}$ *for* $\forall \circ \in O$.

Proof By definition, if $(\widetilde{\mathscr{H}}; O)$ is a multi-group, then for $\forall \circ \in O$, $\widetilde{\mathscr{H}} \cap \mathscr{G}_\circ$ is a group. Whence, $\widetilde{\mathscr{H}} \cap \mathscr{G}_\circ \le \mathscr{G}_\circ^{\max}$.

Conversely, if $\widetilde{\mathscr{H}} \cap \mathscr{G}_\circ \le \mathscr{G}_\circ^{\max}$ for $\forall \circ \in O$, then $\widetilde{\mathscr{H}} \cap \mathscr{G}_\circ$ is a group. Therefore, $\left(\widetilde{\mathscr{H}}; O\right)$ is a multi-group by definition. □

Applying Theorem 3.2.2, we get conclusions following.

Corollary 3.2.1 *An multi-subsystem* $\left(\widetilde{\mathscr{H}}; O\right)$ *of a multi-group* $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ *is a multi-subgroup if and only if* $a \circ b^{-1} \in \widetilde{\mathscr{H}} \cap \mathscr{G}_\circ^{\max}$ *for* $\forall \circ \in O$ *and* $a, b \in \widetilde{\mathscr{H}}$ *provided* $a \circ b$ *existing in* $\left(\widetilde{\mathscr{H}}; O\right)$.

Particularly, if $O = \{\circ\}$, we get a conclusion following.

Corollary 3.2.2 *Let* $\circ \in \widetilde{O}$. *Then* $(\mathscr{H}; \circ)$ *is multi-subgroup of a multi-group* $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ *for* $\mathscr{H} \subset \widetilde{\mathscr{G}}$ *if and only if* $(\mathscr{H}; \circ)$ *is a group, i.e.,* $a \circ b^{-1} \in \mathscr{H}$ *for* $a, b \in \mathscr{H}$.

Corollary 3.2.3 *For a distributed multi-group* $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ *with an operation set* $O\left(\widetilde{G}\right) = \{\times_i | 1 \le i \le n\}$, *a subset* $\widetilde{G_1} \subset \widetilde{G}$ *is a distributed multi-subgroup of* $\widetilde{G}$ *if and only if* $\left(\widetilde{G_1} \bigcap G_k; \times_k\right)$ *is a subgroup of* $(G_k; \times_k)$ *or* $\widetilde{G_1} \bigcap G_k = \emptyset$ *for any integer* k, $1 \le k \le n$.

Proof Clearly, $\widetilde{G_1}$ is a multi-subgroup of $\widetilde{G}$ by Theorem 3.2.2. Furthermore, the distribute laws are true for $\widetilde{G_1}$ because $\widetilde{G_1} \subset \widetilde{G}$ and $O\left(\widetilde{G_1}\right) \subset O\left(\widetilde{G}\right)$. □

For finite multi-subgroups, we get a criterion following.

Theorem 3.2.3 *Let* $\widetilde{G}$ *be a finite multi-group with an operation set* $O\left(\widetilde{G}\right) = \{\times_i | 1 \le i \le n\}$.

A subset $\widetilde{G_1}$ *of* $\widetilde{G}$ *is a multi-subgroup under an operation subset* $O\left(\widetilde{G_1}\right) \subset O\left(\widetilde{G}\right)$ *if and only if* $(\widetilde{G_1}; \times)$ *is closed for each operation "*$\times$*" in* $O\left(\widetilde{G_1}\right)$.

Proof Notice that for a multi-group $\widetilde{G}$, its each multi-subgroup $\widetilde{G_1}$ is complete. Now if $\widetilde{G_1}$ is a complete set under each operation "$\times_i$" in $O\left(\widetilde{G_1}\right)$, we know that $\left(\widetilde{G_1} \bigcap G_i; \times_i\right)$ is a group or an empty set. Whence, we get that

$$\widetilde{G_1} = \bigcup_{i=1}^{n} \left(\widetilde{G_1} \bigcap G_i\right).$$

Therefore, $\widetilde{G_1}$ is a multi-subgroup of $\widetilde{G}$ under the operation set $O(\widetilde{G_1})$. □

For a multi-subgroup $\widetilde{H}$ of multi-group $\widetilde{G}$, $g \in \widetilde{G}$, define

$$g\widetilde{H} = \{g \times h | h \in \widetilde{H}, \times \in O(\widetilde{H})\}.$$

Then for $\forall x, y \in \widetilde{G}$,

$$x\widetilde{H} \bigcap y\widetilde{H} = \emptyset \ \text{ or } \ x\widetilde{H} = y\widetilde{H}.$$

In fact, if $x\widetilde{H} \bigcap y\widetilde{H} \neq \emptyset$, let $z \in x\widetilde{H} \bigcap y\widetilde{H}$, then there exist elements $h_1, h_2 \in \widetilde{H}$ and operations "$\times_i$" and "$\times_j$" such that

$$z = x \times_i h_1 = y \times_j h_2.$$

Since $\widetilde{H}$ is a multi-subgroup, $(\widetilde{H} \bigcap G_i; \times_i)$ is a subgroup. Whence, there exists an inverse element h_1^{-1} in $(\widetilde{H} \bigcap G_i; \times_i)$. We get that

$$x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.$$

i.e.,

$$x = y \times_j h_2 \times_i h_1^{-1}.$$

Whence,

$$x\widetilde{H} \subseteq y\widetilde{H}.$$

Similarly, we can also get that

$$x\widetilde{H} \supseteq y\widetilde{H}.$$

Therefore,

$$x\widetilde{H} = y\widetilde{H}.$$

Denote the union of two set A and B by $A\bigoplus B$ if $A\bigcap B=\emptyset$. The following result is implied in the previous discussion.

Theorem 3.2.4 *For any multi-subgroup $\widetilde{H}$ of a multi-group $\widetilde{G}$, there is a representation set T, $T\subset\widetilde{G}$, such that*

$$\widetilde{G}=\bigoplus_{x\in T}x\widetilde{H}.$$

For the case of finite group, since there is only one binary operation "×" and $|x\widetilde{H}|=|y\widetilde{H}|$ for any $x,y\in\widetilde{G}$, We get a consequence following, which is just the Lagrange theorem for finite groups.

Corollary 3.2.4(Lagrange theorem) *For any finite group G, if H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

A multi-group $(\widetilde{\mathscr{G}};\widetilde{O})$ is said to be a *symmetric n-multi-group* if there are

$$\mathscr{S}_1,\mathscr{S}_2,\cdots,\mathscr{S}_n\subset\widetilde{\mathscr{G}},$$

$\widetilde{O}=\{\circ_i,\ 1\le i\le n\}$ with

(1) $\widetilde{\mathscr{G}}=\bigcup\limits_{i=1}^{n}\mathscr{S}_i$;

(2) $(\mathscr{S}_i;\circ_i)$ is a symmetric group S_{Ω_i} for $1\le i\le n$. We call the n-tuple $(|\Omega_1|,|\Omega_2|,\cdots,|\Omega_n||)$ the *degree of the symmetric n-multi-group* $(\widetilde{\mathscr{G}};\widetilde{O})$.

Now let multi-group $(\widetilde{\mathscr{G}};\widetilde{O})$ be a n-multi-group with $\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_n\subset\widetilde{\mathscr{G}}$, $\widetilde{O}=\{\circ_i,\ 1\le i\le n\}$. For any integer i, $1\le i\le n$, let $\mathscr{G}_{\circ_i}=\{a_{i1}=1_{\mathscr{G}_{\circ_i}},a_{i2},\cdots,a_{in_{\circ_i}}\}$. For $\forall a_{ik}\in\mathscr{G}_{\circ_i}$, define

$$\sigma_{a_{ik}}=\left(\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in}\\ a_{i1}\circ a_{ik} & a_{i2}\circ a_{ik} & \cdots & a_{in_{\circ_i}}\circ a_{ik}\end{array}\right)=\left(\begin{array}{c} a\\ a\circ a_{ik}\end{array}\right),$$

$$\tau_{a_{ik}}=\left(\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in_{\circ_i}}\\ a_{ik}^{-1}\circ a_{i1} & a_{ik}^{-1}\circ a_{i2} & \cdots & a_{ik}^{-1}\circ a_{in_{\circ_i}}\end{array}\right)=\left(\begin{array}{c} a\\ a_{ik}^{-1}\circ a\end{array}\right)$$

Denote by $R_{\mathscr{G}_i}=\{\sigma_{a_{i1}},\sigma_{a_{i2}},\cdots,\sigma_{a_{in_{\circ_i}}}\}$ and $L_{\mathscr{G}_i}=\{\tau_{a_{i1}},\tau_{a_{i2}},\cdots,\tau_{a_{in_{\circ_i}}}\}$ and $\times_i^r$ or $\times_i^l$ the induced multiplication in $R_{\mathscr{G}_i}$ or $L_{\mathscr{G}_i}$. Then we get two sets of permutations

$$R_{\widetilde{\mathscr{G}}}=\bigcup_{i=1}^{n}\{\sigma_{a_{i1}},\sigma_{a_{i2}},\cdots,\sigma_{a_{in_{\circ_i}}}\}\ \text{ and }\ L_{\mathscr{G}}=\bigcup_{i=1}^{n}\{\tau_{a_{i1}},\tau_{a_{i2}},\cdots,\tau_{a_{in_{\circ_i}}}\}.$$

We say $R_{\widetilde{\mathscr{G}}}$, $L_{\widetilde{\mathscr{G}}}$ the *right* or *left regular representation* of $\widetilde{\mathscr{G}}$, respectively. Similar to the *Cayley* theorem, we get the following representation result for multi-groups.

Theorem 3.2.5 *Every multi-group is isomorphic to a multi-subgroup of symmetric multi-group.*

Proof Let multi=group $\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$ be a n-multi=group with $\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_n\subset\widetilde{\mathscr{G}}$, $\widetilde{O}=\{\circ_i,\ 1\le i\le n\}$. For any integer i, $1\le i\le n$. Then $R_{\mathscr{G}_i}$ and $L_{\mathscr{G}_i}$ both are subgroups of the symmetric group $S_{\mathscr{G}_i}$ for any integer $1\le i\le n$. Whence, $(R_{\widetilde{\mathscr{G}}};O^r)$ and $(L_{\widetilde{\mathscr{G}}};O^l)$ both are multi-subgroup of symmetric multi-group by definition, where $O^r=\{\times_i^r|1\le i\le n\}$ and $O^l=\{\times_i^l|1\le i\le n\}$.

We only need to prove that $\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$ is isomorphic to $(R_{\widetilde{\mathscr{G}}};O^r)$. For this objective, define a mapping $(f,\iota):(\widetilde{\mathscr{G}};\widetilde{O})\to(R_{\widetilde{\mathscr{G}}};O^r)$ by

$$f(a_{ik})=\sigma_{a_{ik}}\ \text{ and }\ \iota(\circ_i)=\times_i^r$$

for integers $1\le i\le n$. Such a mapping is one-to-one by definition. It is easily to see that

$$f(a_{ij}\circ_i a_{ik})=\sigma_{a_{ij}\circ_i a_{ik}}=\sigma_{a_{ij}}\times_i^r\sigma_{a_{ik}}=f(a_{ij})\iota(\circ_i)f(a_{ik})$$

for integers $1\le i,k,l\le n$. Whence, (f,ι) is an isomorphism from $(\widetilde{\mathscr{G}};\widetilde{O})$ to $(R_{\widetilde{\mathscr{G}}};O^r)$. Similarly, we can also prove that $\left(\widetilde{\mathscr{G}};\widetilde{O}\right)\simeq(L_{\widetilde{\mathscr{G}}};O^l)$. □

3.2.3 Normal Multi-Subgroup. A multi-subgroup $\left(\widetilde{\mathscr{H}};O\right)$ of $\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$ is *normal*, denoted by $\left(\widetilde{\mathscr{H}};O\right)\lhd\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$ if for $\forall g\in\widetilde{\mathscr{G}}$ and $\forall\circ\in O$, $g\circ\widetilde{\mathscr{H}}=\widetilde{\mathscr{H}}\circ g$, where $g\circ\widetilde{\mathscr{H}}=\{g\circ h|h\in\widetilde{\mathscr{H}}$ provided $g\circ h$ existing$\}$ and $\widetilde{\mathscr{H}}\circ g$ is similarly defined. We get a criterion for normal multi-subgroups of a multi-group following.

Theorem 3.2.6 *Let* $\left(\widetilde{\mathscr{H}};O\right)\le\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$. *Then* $\left(\widetilde{\mathscr{H}};O\right)\lhd\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$ *if and only if*

$$\widetilde{\mathscr{H}}\cap\mathscr{G}_\circ^{\max}\lhd\mathscr{G}_\circ^{\max}$$

for $\forall\circ\in O$.

Proof If $\widetilde{\mathscr{H}}\cap\mathscr{G}_\circ^{\max}\lhd\mathscr{G}_\circ^{\max}$ for $\forall\circ\in O$, then $g\circ\widetilde{\mathscr{H}}=\widetilde{\mathscr{H}}\circ g$ for $\forall g\in\mathscr{G}_\circ^{\max}$ by definition, i.e., all such $g\in\widetilde{\mathscr{G}}$ and $h\in\widetilde{\mathscr{H}}$ with $g\circ h$ and $h\circ g$ defined. So $\left(\widetilde{\mathscr{H}};O\right)\lhd\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$.

Now if $\left(\widetilde{\mathscr{H}};O\right)\lhd\left(\widetilde{\mathscr{G}};\widetilde{O}\right)$, it is clear that $\widetilde{\mathscr{H}}\cap\mathscr{G}_\circ^{\max}\lhd\mathscr{G}_\circ^{\max}$ for $\forall\circ\in O$. □

Corollary 3.2.5 *Let* $\widetilde{G}=\bigcup\limits_{i=1}^{n}G_i$ *be a multi-group with an operation set* $O\left(\widetilde{G}\right)=\{\times_i|1\le i\le n\}$. *Then a multi-subgroup* $\widetilde{H}$ *of* $\widetilde{G}$ *is normal if and only if* $\left(\widetilde{H}\bigcap G_i;\times_i\right)$ *is a normal subgroup of* $(G_i;\times_i)$ *or* $\widetilde{H}\bigcap G_i=\emptyset$ *for any integer* $i, 1\le i\le n$.

For a normal multi-subgroup $\left(\widetilde{\mathscr{H}}; O\right)$ of $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$, we know that

$$\left(a \circ \widetilde{\mathscr{H}}\right) \bigcap \left(b \cdot \widetilde{\mathscr{H}}\right) = \emptyset \ \text{ or } \ a \circ \widetilde{\mathscr{H}} = b \cdot \widetilde{\mathscr{H}}.$$

In fact, if $c \in \left(a \circ \widetilde{\mathscr{H}}\right) \bigcap \left(b \cdot \widetilde{\mathscr{H}}\right)$, then there exists $h_1, h_2 \in \widetilde{\mathscr{H}}$ such that

$$a \circ h_1 = c = b \cdot h_2.$$

So a^{-1} and b^{-1} exist in $\mathscr{G}_{\circ}^{\max}$ and $\mathscr{G}_{\cdot}^{\max}$, respectively. Thus,

$$b^{-1} \cdot a \circ h_1 = b^{-1} \cdot b \cdot h_2 = h_2.$$

Whence,

$$b^{-1} \cdot a = h_2 \circ h_1^{-1} \in \widetilde{\mathscr{H}}.$$

We find that

$$a \circ \widetilde{\mathscr{H}} = b \cdot (h_2 \circ h_1) \circ \widetilde{\mathscr{H}} = b \cdot \widetilde{\mathscr{H}}.$$

This fact enables one to find a partition of $\widetilde{\mathscr{G}}$ following

$$\widetilde{\mathscr{G}} = \bigcup_{g \in \widetilde{\mathscr{G}}, \circ \in \widetilde{O}} g \circ \widetilde{\mathscr{H}}.$$

Choose an element h from each $g \circ \widetilde{\mathscr{H}}$ and denoted by H all such elements, called the *representation* of a partition of $\widetilde{\mathscr{G}}$, i.e.,

$$\widetilde{\mathscr{G}} = \bigcup_{h \in H, \circ \in \widetilde{O}} h \circ \widetilde{\mathscr{H}}.$$

Define the *quotient set* of $\widetilde{\mathscr{G}}$ by $\widetilde{\mathscr{H}}$ to be

$$\widetilde{\mathscr{G}} / \widetilde{\mathscr{H}} = \{h \circ \widetilde{\mathscr{H}} | h \in H, \circ \in O\}.$$

Notice that $\widetilde{\mathscr{H}}$ is normal. We find that

$$\left(a \circ \widetilde{\mathscr{H}}\right) \cdot \left(b \bullet \widetilde{\mathscr{H}}\right) = \widetilde{\mathscr{H}} \circ a \cdot b \bullet \widetilde{\mathscr{H}} = (a \cdot b) \circ \widetilde{\mathscr{H}} \bullet \widetilde{\mathscr{H}} = (a \cdot b) \circ \widetilde{\mathscr{H}}$$

in $\widetilde{\mathscr{G}} / \widetilde{\mathscr{H}}$ for $\circ, \ \bullet, \ \cdot \in \widetilde{O}$, i.e., $(\widetilde{\mathscr{G}} / \widetilde{\mathscr{H}}; O)$ is an algebraic system. It is easily to check that $\left(\widetilde{\mathscr{G}} / \widetilde{\mathscr{H}}; O\right)$ is a multi-group by definition, called the *quotient multi-group* of $\widetilde{\mathscr{G}}$ by $\widetilde{\mathscr{H}}$.

Now let $\left(\widetilde{\mathscr{G}_1}; \widetilde{O_1}\right)$ and $\left(\widetilde{\mathscr{G}_2}; \widetilde{O_2}\right)$ be multi-groups. A mapping pair (ϕ, ι) with $\phi : \widetilde{\mathscr{G}_1} \to \widetilde{\mathscr{G}_2}$ and $\iota : \widetilde{O_1} \to \widetilde{O_2}$ is a *homomorphism* if $\phi(a \circ b) = \phi(a)\iota(\circ)\phi(b)$ for $\forall a, b \in \widetilde{\mathscr{G}}$ and

$\circ \in \widetilde{O}_1$ provided $a \circ b$ existing in $\left(\widetilde{\mathscr{G}}_1; \widetilde{O}_1\right)$. Define the *image* $\mathrm{Im}(\phi, \iota)$ and *kernel* $\mathrm{Ker}(\phi, \iota)$ respectively by

$$\mathrm{Im}(\phi, \iota) = \left\{ \phi(g) \mid g \in \widetilde{\mathscr{G}}_1 \right\},$$
$$\mathrm{Ker}(\phi, \iota) = \left\{ g \mid \phi(g) = 1_{\mathscr{G}_\circ},\ g \in \widetilde{\mathscr{G}}_1\ , \circ \in \widetilde{O}_2 \right\}.$$

Then we get the following isomorphism theorem for multi-groups.

Theorem 3.2.7 *Let* $(\phi, \iota) : \left(\widetilde{\mathscr{G}}_1; \widetilde{O}_1\right) \to \left(\widetilde{\mathscr{G}}_2; \widetilde{O}_2\right)$ *be a homomorphism. Then*

$$\widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota) \simeq \mathrm{Im}(\phi, \iota).$$

Proof Notice that $\mathrm{Ker}(\phi, \iota)$ is a normal multi-subgroup of $\left(\widetilde{\mathscr{G}}_1; \widetilde{O}_1\right)$. We prove that the induced mapping (σ, ω) determined by $(\sigma, \omega) : x \circ \mathrm{Ker}(\phi, \iota) \to \phi(x)$ is an isomorphism from $\widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota)$ to $\mathrm{Im}(\phi, \iota)$.

Now if $(\sigma, \omega)(x_1) = (\sigma, \omega)(x_2)$, then we get that $(\sigma, \omega)(x_1 \circ x_2^{-1}) = 1_{\mathscr{G}_\circ}$ provided $x_1 \circ x_2^{-1}$ existing in $(\widetilde{\mathscr{G}}_1; \widetilde{O}_1)$, i.e., $x_1 \circ x^{-1} \in \mathrm{Ker}(\phi, \iota)$. Thus $x_1 \circ \mathrm{Ker}(\phi, \iota) = x_2 \circ \mathrm{Ker}(\phi, \iota)$, i.e., the mapping (σ, ω) is one-to-one. Whence it is a bijection from $\widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota)$ to $\mathrm{Im}(\phi, \iota)$.

For $\forall a \circ \mathrm{Ker}(\phi, \iota), b \circ \mathrm{Ker}(\phi, \iota) \in \widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota)$ and $\cdot \in \widetilde{O}_1$, we get that

$$\begin{aligned}
&(\sigma, \omega)[a \circ \mathrm{Ker}(\phi, \iota) \cdot b \bullet \mathrm{Ker}(\phi, \iota)]\\
&= (\sigma, \omega)[(a \cdot b) \circ \mathrm{Ker}(\phi, \iota)] = \phi(a \cdot b) = \phi(a)\iota(\cdot)\phi(b)\\
&= (\sigma, \omega)[a \circ \mathrm{Ker}(\phi, \iota)]\iota(\cdot)(\sigma, \omega)[b \bullet \mathrm{Ker}(\phi, \iota)].
\end{aligned}$$

Whence, (σ, ω) is an isomorphism from $\widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota)$ to $\mathrm{Im}(\phi, \iota)$. □

Particularly, let $\left(\widetilde{\mathscr{G}}_2; \widetilde{O}_2\right)$ be a group in Theorem 3.2.7, we get a generalization of the fundamental homomorphism theorem following.

Corollary 3.2.6 *Let* $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ *be a multi-group and* $(\omega, \iota) : \left(\widetilde{\mathscr{G}}; \widetilde{O}\right) \to (\mathscr{A}; \circ)$ *an epimorphism from* $\left(\widetilde{\mathscr{G}}; \widetilde{O}\right)$ *to a group* $(\mathscr{A}; \circ)$*. Then*

$$\widetilde{\mathscr{G}}/\mathrm{Ker}(\omega, \iota) \ \simeq\ (\mathscr{A}; \circ).$$

3.2.4 Multi-Subgroup Series. For a multi-group $\widetilde{G}$ with an operation set $O\left(\widetilde{G}\right) = \{\times_i |\ 1 \leq i \leq n\}$, an order of operations in $O\left(\widetilde{G}\right)$ is said to be an *oriented operation sequence*, denoted by $\overrightarrow{O}\left(\widetilde{G}\right)$. For example, if $O\left(\widetilde{G}\right) = \{\times_1, \times_2 \times_3\}$, then $\times_1 \succ \times_2 \succ \times_3$ is an oriented operation sequence and $\times_2 \succ \times_1 \succ \times_3$ is also an oriented operation sequence.

For a given oriented operation sequence $\overrightarrow{O}\left(\widetilde{G}\right)$, we construct a series of normal multi-subgroups

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_m = \{1_{\times_n}\}$$

by the following programming.

STEP 1: *Construct a series* $\widetilde{G} \rhd \widetilde{G}_{11} \rhd \widetilde{G}_{12} \rhd \cdots \rhd \widetilde{G}_{1l_1}$ *under the operation* "$\times_1$".

STEP 2: *If a series* $\widetilde{G}_{(k-1)l_1} \rhd \widetilde{G}_{k1} \rhd \widetilde{G}_{k2} \rhd \cdots \rhd \widetilde{G}_{kl_k}$ *has be constructed under the operation* "$\times_k$" *and* $\widetilde{G}_{kl_k} \neq \{1_{\times_n}\}$, *then construct a series* $\widetilde{G}_{kl_1} \rhd \widetilde{G}_{(k+1)1} \rhd \widetilde{G}_{(k+1)2} \rhd \cdots \rhd \widetilde{G}_{(k+1)l_{k+1}}$ *under the operation* "$\times_{k+1}$".

This programming is terminated until the series $\widetilde{G}_{(n-1)l_1} \rhd \widetilde{G}_{n1} \rhd \widetilde{G}_{n2} \rhd \cdots \rhd \widetilde{G}_{nl_n} = \{1_{\times_n}\}$ *has be constructed under the operation* "$\times_n$".

The number m is called the *length of the series of normal multi-subgroups*. Call a series of normal multi-subgroups $\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_n = \{1_{\times_n}\}$ *maximal* if there exists a normal multi-subgroup $\widetilde{H}$ for any integer $k, s, 1 \leq k \leq n, 1 \leq s \leq l_k$ such that $\widetilde{G}_{ks} \rhd \widetilde{H} \rhd \widetilde{G}_{k(s+1)}$, then $\widetilde{H} = \widetilde{G}_{ks}$ or $\widetilde{H} = \widetilde{G}_{k(s+1)}$. For a maximal series of finite normal multi-subgroup, we get a result in the following.

Theorem 3.2.8 *For a finite multi-group* $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ *and an oriented operation sequence* $\overrightarrow{O}\left(\widetilde{G}\right)$, *the length of the maximal series of normal multi-subgroup in* $\widetilde{G}$ *is a constant, only dependent on* $\widetilde{G}$ *itself.*

Proof The proof is by the induction principle on the integer n. For $n = 1$, the maximal series of normal multi-subgroups of $\widetilde{G}$ is just a composition series of a finite group. By Jordan-Hölder theorem (see [NiD1] for details), we know the length of a composition series is a constant, only dependent on $\widetilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume that the assertion is true for all cases of $n \leq k$. We prove it is also true in the case of $n = k + 1$. Not loss of generality, assume the order of those binary operations in $\overrightarrow{O}(\widetilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \rhd G_2 \rhd \cdots \rhd G_s = \{1_{\times_1}\}.$$

By Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Corollary 3.2.5, we know a maximal series of normal multi-subgroups of $\widetilde{G}$ gotten by STEP 1 under the operation "$\times_1$" is

$$\widetilde{G} \triangleright \widetilde{G} \setminus (G_1 \setminus G_2) \triangleright \widetilde{G} \setminus (G_1 \setminus G_3) \triangleright \cdots \triangleright \widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\}).$$

Notice that $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is still a multi-group with less or equal to k operations. By the induction assumption, we know the length of the maximal series of normal multi-subgroups in $\widetilde{G}\setminus(G_1\setminus\{1_{\times_1}\})$ is a constant only dependent on $\widetilde{G}\setminus(G_1\setminus\{1_{\times_1}\})$. Therefore, the length of a maximal series of normal multi-subgroups is also a constant, only dependent on $\widetilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal multi-subgroups of $\widetilde{G}$ is a constant under an oriented operations $\overrightarrow{O}(\widetilde{G})$, only dependent on $\widetilde{G}$ itself. □

As a special case of Theorem 3.2.8, we get a consequence following.

Corollary 3.2.7(Jordan-Hölder theorem) *For a finite group G, the length of its composition series is a constant, only dependent on G.*

§3.3 MULTI-RINGS

3.3.1 Multi-Ring. It should be noted that these multi-spaces constructed groups, i.e., distributed multi-groups $\left(\widetilde{G}; O\left(\widetilde{G}\right)\right)$ generalize rings. Similarly, we can also construct multi-spaces by rings or fields.

Definition 3.3.1 *Let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a complete multi-space with a double operation set $O\left(\widetilde{R}\right) = O_1 \bigcup O_2$, where $O_1 = \{\cdot_i, 1 \le i \le m\}$, $O_2 = \{+_i, 1 \le i \le m\}$. If for any integers $i, 1 \le i \le m$, $(R_i; +_i, \cdot_i)$ is a ring, then $\widetilde{R}$ is called a multi-ring, denoted by $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ and $(+_i, \cdot_i)$ a double operation for any integer i. If $(R; +_i, \cdot_i)$ is a skew field or a field for integers $1 \le i \le m$, then $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ is called a skew multi-field or a multi-field.*

For a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, let $\widetilde{S} \subset \widetilde{R}$ and $O_1\left(\widetilde{S}\right) \subset O_1\left(\widetilde{R}\right)$, $O_2\left(\widetilde{S}\right) \subset O_2\left(\widetilde{R}\right)$, if $\widetilde{S}$ is a multi-ring with a double operation set $O\left(\widetilde{S}\right) = O_1\left(\widetilde{S}\right) \bigcup O_2\left(\widetilde{S}\right)$, such a $\widetilde{S}$ is called a *multi-subring* of $\widetilde{R}$.

Theorem 3.3.1 *For a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with $O\left(\widetilde{S}\right) \subset O\left(\widetilde{R}\right)$ is a multi-subring of $\widetilde{R}$ if and only if $\left(\widetilde{S} \bigcap R_k; +_k, \cdot_k\right)$ is a subring of*

$(R_k; +_k, \cdot_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$ for any integer $k, 1 \leq k \leq m$.

Proof For any integer $k, 1 \leq k \leq m$, if $\left(\widetilde{S} \bigcap R_k; +_k, \cdot_k\right)$ is a subring of $(R_k; +_k, \cdot_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$, then $\widetilde{S} = \bigcup\limits_{i=1}^{m} \left(\widetilde{S} \bigcap R_i\right)$ is a multi-subring by definition.

Now if $\widetilde{S} = \bigcup\limits_{j=1}^{s} S_{i_j}$ is a multi-subring of $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with a double operation set $O_1\left(\widetilde{S}\right) = \{\cdot_{i_j}, 1 \leq j \leq s\}$ and $O_2\left(\widetilde{S}\right) = \{+_{i_j}, 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \cdot_{i_j})$ is a subring of $(R_{i_j}; +_{i_j}, \cdot_{i_j})$. Therefore, $S_{i_j} = R_{i_j} \bigcap \widetilde{S}$ for any integer $j, 1 \leq j \leq s$. But $\widetilde{S} \bigcap S_l = \emptyset$ for other integer $l \in \{i; 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$. □

Applying the criterions for subrings of a ring, we get a result for multi-subrings by Theorem 3.3.1 following.

Theorem 3.3.2 *For a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, a subset $\widetilde{S} \subset \widetilde{R}$ with $O\left(\widetilde{S}\right) \subset O\left(\widetilde{R}\right)$ is a multi-subring of $\widetilde{R}$ if and only if $\left(\widetilde{S} \bigcap R_j; +_j\right) \prec (R_j; +_j)$ and $\left(\widetilde{S}; \cdot_j\right)$ is complete for any double operation $(+_j, \cdot_j) \in O\left(\widetilde{S}\right)$.*

Proof According to Theorem 3.3.1, we know that $\widetilde{S}$ is a multi-subring if and only if $\left(\widetilde{S} \bigcap R_i; +_i, \cdot_i\right)$ is a subring of $(R_i; +_i, \cdot_i)$ or $\widetilde{S} \bigcap R_i = \emptyset$ for any integer $i, 1 \leq i \leq m$. By a well known criterion for subrings of a ring (see [NiD1] for details), we know that $\left(\widetilde{S} \bigcap R_i; +_i, \cdot_i\right)$ is a subring of $(R_i; +_i, \cdot_i)$ if and only if $\left(\widetilde{S} \bigcap R_j; +_j\right) \prec (R_j; +_j)$ and $\left(\widetilde{S}; \cdot_j\right)$ is a complete set for any double operation $(+_j, \cdot_j) \in O\left(\widetilde{S}\right)$. □

A multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $O_1 = \{\cdot_i | 1 \leq i \leq l\}, O_2 = \{+_i | 1 \leq i \leq l\}$ is *integral* if for $\forall a, b \in \mathscr{H}$ and an integer i, $1 \leq i \leq l$, $a \cdot_i b = b \cdot_i a$, $1_{\cdot_i} \neq 0_{+i}$ and $a \cdot_i b = 0_{+_i}$ implies that $a = 0_{+_i}$ or $b = 0_{+_i}$. If $l = 1$, an integral l-ring is the integral ring by definition. For the case of multi-rings with finite elements, an integral multi-ring is nothing but a multi-field, such as those shown in the next result.

Theorem 3.3.3 *A finitely integral multi-ring is a multi-field.*

Proof Let $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ be a finitely integral multi-ring with $\widetilde{R} = \{a_1, a_2 \cdots, a_n\}$, where $O_1 = \{\cdot_i | 1 \leq i \leq l\}, O_2 = \{+_i | 1 \leq i \leq l\}$. For any integer $i, 1 \leq i \leq l$, choose an element $a \in \widetilde{R}$ and $a \neq 0_{+_i}$. Then

$$a \cdot_i a_1,\ a \cdot_i a_2, \cdots,\ a \cdot_i a_n$$

are n elements. If $a \cdot_i a_s = a \cdot_i a_t$, i.e., $a \cdot_i (a_s +_i a_t^{-1}) = 0_{+_i}$. By definition, we know that $a_s +_i a_t^{-1} = 0+_i$, namely, $a_s = a_t$. That is, these $a \cdot_i a_1,\ a \cdot_i a_2, \cdots,\ a \cdot_i a_n$ are different two by two. Whence,

$$\widetilde{R} = \{\ a \cdot_i a_1,\ a \cdot_i a_2, \cdots,\ a \cdot_i a_n\ \}.$$

Now assume $a \cdot_i a_s = 1_{\cdot_i}$, then $a^{-1} = a_s$, i.e., each element of $\widetilde{R}$ has an inverse in $(\widetilde{R}; \cdot_i)$, which implies it is a commutative group. Therefore, $\left(\widetilde{R}; +_i, \cdot_i\right)$ is a field for any integer $i, 1 \leq i \leq l$. $\square$

Corollary 3.3.1 *Any finitely integral domain is a field.*

3.3.2 Multi-Ideal. A *multi-ideal* $\widetilde{I}$ of multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ is such a multi-subring of $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ satisfying conditions following:

(1) $\widetilde{I}$ is a multi-subgroup with an operation set $\left\{+ | + \in O_2\left(\widetilde{I}\right)\right\}$;

(2) For any $r \in \widetilde{R}, a \in \widetilde{I}$ and $\times \in O_1\left(\widetilde{I}\right)$, $r \times a \in \widetilde{I}$ and $a \times r \in \widetilde{I}$ provided all of these operating results exist.

Theorem 3.3.4 *A subset $\widetilde{I}$ with $O_1\left(\widetilde{I}\right) \subset O_1\left(\widetilde{R}\right)$, $O_2\left(\widetilde{I}\right) \subset O_2(\widetilde{R})$ of a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, $O_1\left(\widetilde{R}\right) = \{\times_i |\ 1 \leq i \leq m\}$ and $O_2\left(\widetilde{R}\right) = \{+_i |\ 1 \leq i \leq m\}$ is a multi-ideal if and only if $\left(\widetilde{I} \bigcap R_i, +_i, \times_i\right)$ is an ideal of ring $(R_i, +_i, \times_i)$ or $\widetilde{I} \bigcap R_i = \emptyset$ for any integer $i, 1 \leq i \leq m$.*

Proof By the definition of multi-ideal, the necessity of these conditions is obvious.

For the sufficiency, denote by $\widetilde{R}(+, \times)$ the set of elements in $\widetilde{R}$ with binary operations "+" and "×". If there exists an integer i such that $\widetilde{I} \bigcap R_i \neq \emptyset$ and $\left(\widetilde{I} \bigcap R_i, +_i, \times_i\right)$ is an ideal of $(R_i, +_i, \times_i)$, then for $\forall a \in \widetilde{I} \bigcap R_i$, $\forall r_i \in R_i$, we know that

$$r_i \times_i a \in \widetilde{I} \bigcap R_i; \quad a \times_i r_i \in \widetilde{I} \bigcap R_i.$$

Notice that $\widetilde{R}(+_i, \times_i) = R_i$. Therefore, we get that

$$r \times_i a \in \widetilde{I} \bigcap R_i \text{ and } a \times_i r \in \widetilde{I} \bigcap R_i,$$

for $\forall r \in \widetilde{R}$ provided all of these operating results exist. Whence, $\widetilde{I}$ is a multi-ideal of $\widetilde{R}$. $\square$

3.3.3 Multi-Ideal Chain. A multi-ideal $\widetilde{I}$ of a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ is said to be *maximal* if for any multi-ideal $\widetilde{I}'$, $\widetilde{R} \supseteq \widetilde{I}' \supseteq \widetilde{I}$ implies that $\widetilde{I}' = \widetilde{R}$ or $\widetilde{I}' = \widetilde{I}$. For an order of the double operations in $O\left(\widetilde{R}\right)$ of a multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$, not loss of generality, let it to be $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$, we can define a *multi-ideal chain* of $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ by the following programming.

(1) Construct a multi-ideal chain $\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1}$ under the double operation $(+_1, \times_1)$, where $\widetilde{R}_{11}$ is a maximal multi-ideal of $\widetilde{R}$ and in general, $\widetilde{R}_{1(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{1i}$ for any integer i, $1 \leq i \leq m-1$.

(2) If a multi-ideal chain $\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i}$ has been constructed for $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_i, \times_i)$, $1 \leq i \leq m-1$, then construct a multi-ideal chain of $\widetilde{R}_{is_i}$ by $\widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \widetilde{R}_{(i+1)2} \supset \cdots \supset \widetilde{R}_{(i+1)s_1}$ under the double operation $(+_{i+1}, \times_{i+1})$, where $\widetilde{R}_{(i+1)1}$ is a maximal multi-ideal of $\widetilde{R}_{is_i}$ and in general, $\widetilde{R}_{(i+1)(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define a multi-ideal chain of $\widetilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$ to be $\widetilde{R} \supset \widetilde{R}_{11} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \cdots \supset \widetilde{R}_{(i+1)s_{i+1}}$.

We get a result on multi-ideal chains of multi-rings following.

Theorem 3.3.5 *For a multi-ring* $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ *with* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, *its multi-ideal chain has finite terms if and only if the ideal chain of ring* $(R_i; +_i, \times_i)$ *has finite terms, i.e., each ring* $(R_i; +_i, \times_i)$ *is an Artin ring for any integer* $i, 1 \leq i \leq m$.

Proof Let

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

be the order of these double operations in $\overrightarrow{O}\left(\widetilde{R}\right)$ and let

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}$$

be a maximal ideal chain in ring $(R_1; +_1, \times_1)$. Calculation shows that

$$\widetilde{R}_{11} = \widetilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\widetilde{R}_{12} = \widetilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\widetilde{R}_{1t_1} = \widetilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m} R_i).$$

According to Theorem 3.3.4, we know that

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1t_1}$$

is a maximal multi-ideal chain of $\widetilde{R}$ under the double operation $(+_1, \times_1)$. In general, for any integer $i, 1 \leq i \leq m-1$, we assume that

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in ring $(R_{(i-1)t_{i-1}};+_i,\times_i)$. Calculation shows that

$$\widetilde{R}_{ik}=R_{ik}\bigcup(\bigcup_{j=i+1}^{m}\widetilde{R}_{ik}\bigcap R_i).$$

Then we know that

$$\widetilde{R}_{(i-1)t_{i-1}}\supset\widetilde{R}_{i1}\supset\widetilde{R}_{i2}\supset\cdots\supset\widetilde{R}_{it_i}$$

is a maximal multi-ideal chain of $\widetilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i,\times_i)$ by Theorem 3.3.4. Whence, if the ideal chain of ring $(R_i;+_i,\times_i)$ has finite terms for any integer $i, 1\leq i\leq m$, then the multi-ideal chain of multi-ring $\widetilde{R}$ only has finite terms. Now if there exists an integer i_0 such that the ideal chain of ring $(R_{i_0},+_{i_0},\times_{i_0})$ has infinite terms, then there must also be infinite terms in a multi-ideal chain of multi-ring $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$. □

A multi-ring is called an *Artin multi-ring* if its each multi-ideal chain only has finite terms. We get a consequence following by Theorem 3.3.5.

Corollary 3.3.2 *A multi-ring* $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$ *with* $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ *and a double operation set* $O\left(\widetilde{R}\right)=\{(+_i,\times_i)|\ 1\leq i\leq m\}$ *is an Artin multi-ring if and only if each ring* $(R_i;+_i,\times_i)$ *is an Artin ring for integers* $i, 1\leq i\leq m$.

For a multi-ring $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$ with $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ and double operation set $O\left(\widetilde{R}\right)=\{(+_i,\times_i)|\ 1\leq i\leq m\}$, an element e is an *idempotent* element if $e^2_{\times}=e\times e=e$ for a double binary operation $(+,\times)\in O\left(\widetilde{R}\right)$. Define the *directed sum* $\widetilde{I}$ of two multi-ideals $\widetilde{I}_1,\widetilde{I}_2$ by

(1) $\widetilde{I}=\widetilde{I}_1\bigcup\widetilde{I}_2$;

(2) $\widetilde{I}_1\bigcap\widetilde{I}_2=\{0_+\}$, or $\widetilde{I}_1\bigcap\widetilde{I}_2=\emptyset$, where 0_+ denotes the unit under the operation +.

Such a directed sum of $\widetilde{I}_1,\widetilde{I}_2$ is usually denote by

$$\widetilde{I}=\widetilde{I}_1\bigoplus\widetilde{I}_2.$$

Now if $\widetilde{I}=\widetilde{I}_1\bigoplus\widetilde{I}_2$ for any $\widetilde{I}_1,\widetilde{I}_2$ implies that $\widetilde{I}_1=\widetilde{I}$ or $\widetilde{I}_2=\widetilde{I}$, then $\widetilde{I}$ is called *non-reducible*. We get the following result for Artin multi-rings.

Theorem 3.3.6 *Every Artin multi-ring* $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$ *with* $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ *and a double operation set* $O\left(\widetilde{R}\right)=\{(+_i,\times_i)|\ 1\leq i\leq m\}$ *is a directed sum of finite non-reducible multi-ideals, and if* $(R_i;+_i,\times_i)$ *has unit* $1_{\times_i}$ *for any integer* $i, 1\leq i\leq m$, *then*

$$\widetilde{R}=\bigoplus_{i=1}^{m}(\bigoplus_{j=1}^{s_i}(R_i\times_i e_{ij})\bigcup(e_{ij}\times_i R_i)),$$

where $e_{ij}, 1 \leq j \leq s_i$ *are orthogonal idempotent elements of ring* $(R_i; +_i, \times_i)$.

Proof Denote by $\widetilde{M}$ the set of multi-ideals which can not be represented by a directed sum of finite multi-ideals in $\widetilde{R}$. According to Theorem 3..3.5, there is a minimal multi-ideal $\widetilde{I_0}$ in $\widetilde{M}$. It is obvious that $\widetilde{I_0}$ is reducible.

Assume that $\widetilde{I_0} = \widetilde{I_1} + \widetilde{I_2}$. Then $\widetilde{I_1} \notin \widetilde{M}$ and $\widetilde{I_2} \notin \widetilde{M}$. Therefore, $\widetilde{I_1}$ and $\widetilde{I_2}$ can be represented by a directed sum of finite multi-ideals. Thereby $\widetilde{I_0}$ can be also represented by a directed sum of finite multi-ideals, contradicts to that $\widetilde{I_0} \in \widetilde{M}$.

Now let

$$\widetilde{R} = \bigoplus_{i=1}^{s} \widetilde{I_i},$$

where each $\widetilde{I_i}, 1 \leq i \leq s$ is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible multi-ideal of $\widetilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \quad e \in R(\times).$$

Whence, there is a set $T \subset \widetilde{R}$ such that

$$\widetilde{R} = \bigoplus_{e \in T,\ \times \in O(\widetilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

Now let $1_\times$ be the unit for an operation $\times \in O(\widetilde{R})$. Assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_l,\ e_i \in T,\ 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \text{ and } e_i \times e_j = 0_i \text{ for } i \neq j.$$

That is, $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of $\widetilde{R}(\times)$. Notice that $\widetilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +_h, \times_h)$. Denote by e_{hi} for e_i, $1 \leq i \leq l$. Consider all units in $\widetilde{R}$, we get that

$$\widetilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)).$$

This completes the proof. □

Corollary 3.3.3 *Every Artin ring* $(R;+,\times)$ *is a directed sum of finite ideals, and if* $(R\,;+,\times)$ *has unit* $1_\times$, *then*

$$R = \bigoplus_{i=1}^{s} R_i e_i,$$

where $e_i, 1 \le i \le s$ *are orthogonal idempotent elements of ring* $(R;+,\times)$.

§3.4 VECTOR MULTI-SPACES

3.4.1 Vector Multi-Space. Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-space with an operation set $O\left(\widetilde{V}\right) = \{(\dot{+}_i, \cdot_i) \mid 1 \le i \le m\}$ and let $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ be a multi-filed with a double operation set $O\left(\widetilde{F}\right) = \{(+_i, \times_i) \mid 1 \le i \le k\}$. If for any integers i, $1 \le i \le k$, $(V_i; F_i)$ is a vector space on F_i with vector additive "$\dot{+}_i$" and scalar multiplication "$\cdot_i$", then $\widetilde{V}$ is called a vector multi-space on the multi-filed $\widetilde{F}$, denoted by $\left(\widetilde{V}; \widetilde{F}\right)$.

For subsets $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, if $\left(\widetilde{V}_1; \widetilde{F}_1\right)$ is also a vector multi-space, then we call $\left(\widetilde{V}_1; \widetilde{F}_1\right)$ a *vector multi-subspace* of $\left(\widetilde{V}; \widetilde{F}\right)$. Similar to the linear spaces, we get the following criterion for vector multi-subspaces.

Theorem 3.4.1 *For a vector multi-space* $\left(\widetilde{V}; \widetilde{F}\right)$, $\widetilde{V}_1 \subset \widetilde{V}$ *and* $\widetilde{F}_1 \subset \widetilde{F}$, $\left(\widetilde{V}_1; \widetilde{F}_1\right)$ *is a vector multi-subspace of* $(\widetilde{V}; \widetilde{F})$ *if and only if for any vector additive* "$\dot{+}$", *scalar multiplication* "$\cdot$" *in* $\left(\widetilde{V}_1; \widetilde{F}_1\right)$ *and* $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$,

$$\alpha \cdot \mathbf{a} \dot{+} \mathbf{b} \in \widetilde{V}_1$$

provided these operating results exist.

Proof Denote by $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$, $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Notice that $\widetilde{V}_1 = \bigcup\limits_{i=1}^{k} \left(\widetilde{V}_1 \bigcap V_i\right)$. By definition, we know that $\left(\widetilde{V}_1; \widetilde{F}_1\right)$ is a vector multi-subspace of $\left(\widetilde{V}; \widetilde{F}\right)$ if and only if for any integer $i, 1 \le i \le k$, $\left(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i\right)$ is a vector subspace of $(V_i, +_i, \cdot_i)$ and $\widetilde{F}_1$ is a multi-filed subspace of $\widetilde{F}$ or $\widetilde{V}_1 \bigcap V_i = \emptyset$.

By Theorem 1.4.1, we know that $\left(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i\right)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ for any integer $i, 1 \le i \le k$ if and only if for $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}_1 \bigcap V_i$, $\alpha \in F_i$,

$$\alpha \cdot_i \mathbf{a} \dot{+}_i \mathbf{b} \in \widetilde{V}_1 \bigcap V_i.$$

That is, for any vector additive "$\dot{+}$", scalar multiplication "$\cdot$" in $(\widetilde{V}_1; \widetilde{F}_1)$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$, if $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b}$ exists, then $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b} \in \widetilde{V}_1$. $\square$

Corollary 3.4.1 *Let* $\left(\widetilde{U};\widetilde{F_1}\right),\left(\widetilde{W};\widetilde{F_2}\right)$ *be two vector multi-subspaces of a vector multi-space* $\left(\widetilde{V};\widetilde{F}\right)$*. Then* $\left(\widetilde{U}\bigcap\widetilde{W};\widetilde{F_1}\bigcap\widetilde{F_2}\right)$ *is a vector multi-space.*

3.4.2 Basis. For a vector multi-space $\left(\widetilde{V};\widetilde{F}\right)$, vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n\in\widetilde{V}$, if there are scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$ such that

$$\alpha_1\cdot_1\mathbf{a}_1\dotplus_1\alpha_2\cdot_2\mathbf{a}_2\dotplus_2\cdots\dotplus_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}\in\widetilde{V}$ is the unit under an operation "+" in $\widetilde{V}$ and $\dotplus_i,\cdot_i\in O\left(\widetilde{V}\right)$, then these vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ are said to be *linearly dependent*. Otherwise, vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ are said to be *linearly independent*.

Notice that there are two cases for linearly independent vectors $\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n$ in a vector multi-space:

(1) For scalars $\alpha_1,\alpha_2,\cdots,\alpha_n\in\widetilde{F}$, if

$$\alpha_1\cdot_1\mathbf{a}_1\dotplus_1\alpha_2\cdot_2\mathbf{a}_2\dotplus_2\cdots\dotplus_{n-1}\alpha_n\cdot_n\mathbf{a}_n=\mathbf{0},$$

where $\mathbf{0}$ is the unit of $\widetilde{V}$ under an operation "+"in $O\left(\widetilde{V}\right)$, then $\alpha_1=0_{+_1},\alpha_2=0_{+_2},\cdots,\alpha_n=0_{+_n}$, where 0_{+_i} is the unit under the operation "$+_i$" in $\widetilde{F}$ for integer i, $1\leq i\leq n$.

(2) The operating result of $\alpha_1\cdot_1\mathbf{a}_1\dotplus_1\alpha_2\cdot_2\mathbf{a}_2\dotplus_2\cdots\dotplus_{n-1}\alpha_n\cdot_n\mathbf{a}_n$ does not exist in $\left(\widetilde{V};\widetilde{F}\right)$.

Now for a subset $\widehat{S}\subset\widetilde{V}$, define its *linearly spanning set* $\left\langle\widehat{S}\right\rangle$ by

$$\left\langle\widehat{S}\right\rangle=\{\,\mathbf{a}\mid\mathbf{a}=\alpha_1\cdot_1\mathbf{a}_1\dotplus_1\alpha_2\cdot_2\mathbf{a}_2\dotplus_2\cdots\in\widetilde{V},\mathbf{a}_i\in\widehat{S},\alpha_i\in\widetilde{F},i\geq 1\}.$$

For a vector multi-space $\left(\widetilde{V};\widetilde{F}\right)$, if there exists a subset $\widehat{S},\widehat{S}\subset\widetilde{V}$ such that $\widetilde{V}=\left\langle\widehat{S}\right\rangle$, then we say $\widehat{S}$ is a *linearly spanning set* of the vector multi-space $\widetilde{V}$. If these vectors in a linearly spanning set $\widehat{S}$ of vector multi-space $\widetilde{V}$ are linearly independent, then $\widehat{S}$ is said to be a *basis* of $\left(\widetilde{V};\widetilde{F}\right)$.

Theorem 3.4.2 *A vector multi-space* $\left(\widetilde{V};\widetilde{F}\right)$ *with* $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i,\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$ *has a basis if each vector space* $(V_i;F_i)$ *has a basis for integers* $1\leq i\leq n$.

Proof Let $\Delta_i=\{\mathbf{a}_{i1},\mathbf{a}_{i2},\cdots,\mathbf{a}_{in_i}\}$ be a basis of vector space $(V_i;F_i)$ for $1\leq i\leq k$. Define

$$\widehat{\Delta}=\bigcup_{i=1}^{k}\Delta_i.$$

Then $\widehat{\Delta}$ is a linearly spanning set for $\widetilde{V}$ by definition.

If these vectors in $\widehat{\Delta}$ are linearly independent, then $\widehat{\Delta}$ is a basis of $\widetilde{V}$. Otherwise, choose a vector $\mathbf{b}_1 \in \widehat{\Delta}$ and define $\widehat{\Delta}_1 = \widehat{\Delta} \setminus \{\mathbf{b}_1\}$.

If we have obtained a set $\widehat{\Delta}_s$, $s \geq 1$ and it is not a basis, choose a vector $\mathbf{b}_{s+1} \in \widehat{\Delta}_s$ and define $\widehat{\Delta}_{s+1} = \widehat{\Delta}_s \setminus \{\mathbf{b}_{s+1}\}$.

If these vectors in $\widehat{\Delta}_{s+1}$ are linearly independent, then $\widehat{\Delta}_{s+1}$ is a basis of $\widetilde{V}$. Otherwise, we can define a set $\widehat{\Delta}_{s+2}$ again. Continue this process. Notice that all vectors in Δ_i are linearly independent for any integer $i, 1 \leq i \leq k$. Thus we finally get a basis of $\widetilde{V}$. $\square$

A multi-vector space $\widetilde{V}$ is *finite-dimensional* if it has a finite basis. By Theorem 3.4.2, if the vector space $(V_i; F_i)$ is finite-dimensional for any integer $i, 1 \leq i \leq k$, then $\left(\widetilde{V}; \widetilde{F}\right)$ is finite-dimensional. On the other hand, if there is an integer $i_0, 1 \leq i_0 \leq k$ such that the vector space $(V_{i_0}; F_{i_0})$ is infinite-dimensional, then $\left(\widetilde{V}; \widetilde{F}\right)$ must be infinite-dimensional. This fact enables one to get a consequence following.

Corollary 3.4.2 *Let* $\left(\widetilde{V}; \widetilde{F}\right)$ *be a vector multi-space with* $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. *Then* $\left(\widetilde{V}; \widetilde{F}\right)$ *is finite-dimensional if and only if* $(V_i; +_i, \cdot_i)$ *is finite-dimensional for any integer* $i, 1 \leq i \leq k$.

Furthermore, we know the following result on finite-dimensional multi-spaces.

Theorem 3.4.3 *For a finite-dimensional multi-vector space* $\left(\widetilde{V}; \widetilde{F}\right)$, *any two bases have the same number of vectors.*

Proof Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. The proof is by the induction on k. For $k = 1$, the assertion is true by Corollary 1.4.1.

If $k = 2$, let W_1, W_2 be two subspaces of a finite-dimensional vector space. By Theorem 1.4.5 if the basis of $W_1 \bigcap W_2$ is $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t\}$, then the basis of $W_1 \bigcup W_2$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\},$$

where $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}\}$ is a basis of W_1 and $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}$ a basis of W_2.

Whence, if $\widetilde{V} = W_1 \bigcup W_2$ and $\widetilde{F} = F_1 \bigcup F_2$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}.$$

Now assume the assertion is true for $k = l, l \geq 2$. We consider the case of $k = l + 1$.

Notice that

$$\widetilde{V}=\left(\bigcup_{i=1}^{l}V_i\right)\bigcup V_{l+1},\ \widetilde{F}=\left(\bigcup_{i=1}^{l}F_i\right)\bigcup F_{l+1}.$$

By the induction assumption, we know that any two bases of the multi-vector space $\left(\bigcup\limits_{i=1}^{l}V_i;\bigcup\limits_{i=1}^{l}F_i\right)$ have the same number p of vectors. If the basis of $\left(\bigcup\limits_{i=1}^{l}V_i\right)\bigcap V_{l+1}$ is $\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n\}$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n,\mathbf{f}_{n+1},\mathbf{f}_{n+2},\cdots,\mathbf{f}_p,\mathbf{g}_{n+1},\mathbf{g}_{n+2},\cdots,\mathbf{g}_{dimV_{l+1}}\},$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n,\mathbf{f}_{n+1},\mathbf{f}_{n+2},\cdots,\mathbf{f}_p\}$ is a basis of $\left(\bigcup\limits_{i=1}^{l}V_i;\bigcup\limits_{i=1}^{l}F_i\right)$ and $\{\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n,\mathbf{g}_{n+1},\mathbf{g}_{n+2},\cdots,\mathbf{g}_{dimV_{l+1}}\}$ is a basis of V_{l+1}. Whence, the number of vectors in a basis of $\widetilde{V}$ is $p+dimV_{l+1}-n$ for the case $n=l+1$.

Therefore, the assertion is true for any integer k by the induction principle. □

3.4.3 Dimension. By Theorem 3.4.3, the cardinal number in a basis of a finite dimensional vector multi-space $\widetilde{V}$ is defined to be its *dimension* and denoted by $\dim\widetilde{V}$.

Theorem 3.4.4(*dimensional formula*) *For a vector multi-space* $\left(\widetilde{V};\widetilde{F}\right)$ *with* $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i$ *and* $\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$, *the dimension* $\dim\widetilde{V}$ *of* $\left(\widetilde{V};\widetilde{F}\right)$ *is*

$$\dim\widetilde{V}=\sum_{i=1}^{k}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}}\dim\left(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\right).$$

Proof The proof is by induction on k. If $k=1$, the formula is turn to a trivial case $\dim\widetilde{V}=\dim V_1$. If $k=2$, the formula is

$$\dim\widetilde{V}=\dim V_1+\dim V_2-\dim\left(V_1\bigcap\dim V_2\right),$$

which is true by Theorem 1.4.5.

Now assume that the formula is true for $k=n$. Consider the case of $k=n+1$. According to Theorem 3.4.3, we know that

$$\begin{aligned}\dim\widetilde{V}&=\dim\left(\bigcup_{i=1}^{n}V_i\right)+\dim V_{n+1}-\dim\left(\left(\bigcup_{i=1}^{n}V_i\right)\bigcap V_{n+1}\right)\\&=\dim\left(\bigcup_{i=1}^{n}V_i\right)+\dim V_{n+1}-\dim\left(\bigcup_{i=1}^{n}\left(V_i\bigcap V_{n+1}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \dim V_{n+1} + \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}}\dim\left(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\right)\\
&+ \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}}\dim\left(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\bigcap V_{n+1}\right)\\
&= \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}}\dim\left(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\right).
\end{aligned}
$$

By the induction principle, the formula is true for any integer k. □

As a consequence, we get the following formula.

Corollary 3.4.3(*additive formula*) *For any two vector multi-spaces $\widetilde{V}_1, \widetilde{V}_2$,*

$$\dim(\widetilde{V}_1\bigcup\widetilde{V}_2) = \dim\widetilde{V}_1 + \dim\widetilde{V}_2 - \dim\left(\widetilde{V}_1\bigcap\widetilde{V}_2\right).$$

§3.5 MULTI-MODULES

3.5.1 Multi-Module. The multi-modules are generalization of vector multi-spaces. Let $O = \{ +_i \mid 1 \leq i \leq m\}$, $O_1 = \{\cdot_i | 1 \leq i \leq m\}$ and $O_2 = \{\dot{+}_i | 1 \leq i \leq m\}$ be operation sets, $(\mathscr{M}; O)$ a commutative m-group with units 0_{+_i} and $(\mathscr{R}; O_1 \hookrightarrow O_2)$ a multi-ring with a unit $1_\cdot$ for $\forall \cdot \in O_1$. For any integer i, $1 \leq i \leq m$, define a binary operation $\times_i : \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ by $a \times_i x$ for $a \in \mathscr{R}$, $x \in \mathscr{M}$ such that for $\forall a, b \in \mathscr{R}$, $\forall x, y \in \mathscr{M}$, conditions following hold:

(1) $a \times_i (x +_i y) = a \times_i x +_i a \times_i y$;

(2) $(a \dot{+}_i b) \times_i x = a \times_i x +_i b \times_i x$;

(3) $(a \cdot_i b) \times_i x = a \times_i (b \times_i x)$;

(4) $1_{\cdot_i} \times_i x = x$.

Then $(\mathscr{M}; O)$ is said an *algebraic multi-module over* $(\mathscr{R}; O_1 \hookrightarrow O_2)$ abbreviated to an *m-module* and denoted by **Mod**$(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$. In the case of $m = 1$, It is obvious that **Mod**$(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ is a *module*, particularly, if $(\mathscr{R}; O_1 \hookrightarrow O_2)$ is a field, then **Mod**$(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ is a *linear space* in classical algebra.

For any integer k, $a_i \in \mathscr{R}$ and $x_i \in \mathscr{M}$, where $1 \leq i, k \leq s$, equalities following are hold by induction on the definition of m-modules.

$$a \times_k (x_1 +_k x_2 +_k \cdots +_k x_s) = a \times_k x_1 +_k a \times_k x_2 +_k \cdots +_k a_s \times_k x,$$

$$(a_1 \dot{+}_k a_2 \dot{+}_k \cdots \dot{+}_k a_s) \times_k x = a_1 \times_k x +_k a_2 \times_k x +_k \cdots +_k a_s \times_k x,$$

$$(a_1 \cdot_k a_2 \cdot_k \cdots \cdot_k a_s) \times_k x = a_1 \times_k (a_2 \times_k \cdots \times (a_s \times_k x) \cdots)$$

and

$$1_{\cdot_{i_1}} \times_{i_1} (1_{\cdot_{i_2}} \times_{i_2} \cdots \times_{i_{s-1}} (1_{\cdot_{i_s}} \times_{i_s} x) \cdots) \; = \; x$$

for integers $i_1, i_2, \cdots, i_s \in \{1, 2, \cdots, m\}$.

Notice that for $\forall a, x \in \mathscr{M}$, $1 \le i \le m$,

$$a \times_i x = a \times_i (x +_i 0_{+_i}) = a \times_i x +_i a \times_i 0_{+_i},$$

we find that $a \times_i 0_{+_i} = 0_{+_i}$. Similarly, $0_{\dot{+}_i} \times_i a = 0_{+_i}$. Applying this fact, we know that

$$a \times_i x +_i a^-_{\dot{+}_i} \times_i x = (a \dot{+}_i a^-_{\dot{+}_i}) \times_i x = 0_{\dot{+}_i} \times_i x = 0_{+_i}$$

and

$$a \times_i x +_i a \times_i x^-_{+_i} = a \times_i (x +_i x^-_{+_i}) = a \times_i 0_{+_i} = 0_{+_i}.$$

We know that

$$(a \times_i x)^-_{+_i} = a^-_{\dot{+}_i} \times_i x = a \times_i x^-_{+_i}.$$

Notice that $a \times_i x = 0_{+_i}$ does not always mean $a = 0_{\dot{+}_i}$ or $x = 0_{+_i}$ in an m-module $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ unless $a^-_{\dot{+}_i}$ is existing in $(\mathscr{R}; \dot{+}_i, \cdot_i)$ if $x \neq 0_{+_i}$.

Now choose $\mathbf{Mod}(\mathscr{M}_1(O_1) : \mathscr{R}_1(O_1^1 \hookrightarrow O_2^1))$ an m-module with operation sets $O_1 = \{ +'_i \mid 1 \le i \le m\}$, $O_1^1 = \{\cdot_i^1 | 1 \le i \le m\}$, $O_2^1 = \{\dot{+}_i^1 | 1 \le i \le m\}$ and $\mathbf{Mod}(\mathscr{M}_2(O_2) : \mathscr{R}_2(O_1^2 \hookrightarrow O_2^2))$ an n-module with operation sets $O_2 = \{ +''_i \mid 1 \le i \le n\}$, $O_1^2 = \{\cdot_i^2 | 1 \le i \le n\}$, $O_2^2 = \{\dot{+}_i^2 | 1 \le i \le n\}$. They are said *homomorphic* if there is a mapping $\iota : \mathscr{M}_1 \to \mathscr{M}_2$ such that for any integer i, $1 \le i \le m$,

(1) $\iota(x +'_i y) = \iota(x) +'' \iota(y)$ for $\forall x, y \in \mathscr{M}_1$, where $\iota(+'_i) = +'' \in O_2$;

(2) $\iota(a \times_i x) = a \times_i \iota(x)$ for $\forall x \in \mathscr{M}_1$.

If ι is a bijection, these modules $\mathbf{Mod}(\mathscr{M}_1(O_1) : \mathscr{R}_1(O_1^1 \hookrightarrow O_2^1))$ and $\mathbf{Mod}(\mathscr{M}_2(O_2) : \mathscr{R}_2(O_1^2 \hookrightarrow O_2^2))$ are said to be *isomorphic*, denoted by

$$\mathbf{Mod}(\mathscr{M}_1(O_1) : \mathscr{R}_1(O_1^1 \hookrightarrow O_2^1)) \simeq \mathbf{Mod}(\mathscr{M}_2(O_2) : \mathscr{R}_2(O_1^2 \hookrightarrow O_2^2)).$$

Let $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ be an m-module. For a multi-subgroup $(\mathscr{N}; O)$ of $(\mathscr{M}; O)$, if for any integer i, $1 \le i \le m$, $a \times_i x \in \mathscr{N}$ for $\forall a \in \mathscr{R}$ and $x \in \mathscr{N}$, then by definition it is itself an m-module, called a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$.

Now if $\mathbf{Mod}(\mathscr{N}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$, by Theorem 2.3.2, we can get a quotient multi-group $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ with a representation pair $(R, \widetilde{P})$ under operations

$$(a +_i \mathscr{N}) + (b +_j \mathscr{N}) = (a + b) +_i \mathscr{N}$$

for $\forall a, b \in R, + \in \mathcal{O}$. For convenience, we denote elements $x +_i \mathscr{N}$ in $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ by $\overline{x^{(i)}}$. For an integer i, $1 \leq i \leq m$ and $\forall a \in \mathscr{R}$, define

$$a \times_i \overline{x^{(i)}} = \overline{(a \times_i x)^{(i)}}.$$

Then it can be shown immediately that

(1) $a \times_i (\overline{x^{(i)}} +_i \overline{y^{(i)}}) = a \times_i \overline{x^{(i)}} +_i a \times_i \overline{y^{(i)}}$;

(2) $(a \dot{+}_i b) \times_i \overline{x^{(i)}} = a \times_i \overline{x^{(i)}} +_i b \times_i \overline{x^{(i)}}$;

(3) $(a \cdot_i b) \times_i \overline{x^{(i)}} = a \times_i (b \times_i \overline{x^{(i)}})$;

(4) $1_{\cdot_i} \times_i \overline{x^{(i)}} = \overline{x^{(i)}}$,

i.e.,$(\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})} : \mathscr{R})$ is also an m-module, called a quotient module of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ to $\mathbf{Mod}(\mathscr{N}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$. Denoted by $\mathbf{Mod}(\mathscr{M}/\mathscr{N})$.

The result on homomorphisms of m-modules following is an immediately consequence of Theorem 3.2.7.

Theorem 3.5.1 *Let* $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1))$, $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ *be multi-modules with* $\mathcal{O}_1 = \{ +_i' \mid 1 \leq i \leq m\}$, $\mathcal{O}_2 = \{ +_i'' \mid 1 \leq i \leq n\}$, $\mathcal{O}_1^1 = \{\cdot_i^1 | 1 \leq i \leq m\}$, $\mathcal{O}_2^1 = \{\dot{+}_i^1 | 1 \leq i \leq m\}$, $\mathcal{O}_1^2 = \{\cdot_i^2 | 1 \leq i \leq n\}$, $\mathcal{O}_2^2 = \{\dot{+}_i^2 | 1 \leq i \leq n\}$ *and* $\iota : \mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)) \to \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ *be a onto homomorphism with* $(\mathcal{I}(\mathcal{O}_2); \mathcal{O}_2)$ *a multi-group, where* $\mathcal{I}(\mathcal{O}_2^2)$ *denotes all units in the commutative multi-group* $(\mathscr{M}_2; \mathcal{O}_2)$. *Then there exist representation pairs* $(R_1, \widetilde{P}_1), (R_2, \widetilde{P}_2)$ *such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)} \simeq \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))|_{(R_2,\widetilde{P}_2)},$$

where $\mathscr{N} = \mathrm{Ker}\iota$ *is the kernel of* ι. *Particularly, if* $(\mathcal{I}(\mathcal{O}_2); \mathcal{O}_2)$ *is trivial, i.e.,* $|\mathcal{I}(\mathcal{O}_2)| = 1$, *then*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)} \simeq \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))|_{(R_2,\widetilde{P}_2)}.$$

Proof Notice that $(\mathcal{I}(O_2); O_2)$ is a commutative multi-group. We can certainly construct a quotient module $\mathbf{Mod}(\mathscr{M}_2(O_2)/\mathcal{I}(O_2))$. Applying Theorem 3.2.7, we find that

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)} \simeq \mathbf{Mod}(\mathscr{M}_2(O_2)/\mathcal{I}(O_2))|_{(R_2,\widetilde{P}_2)}.$$

Notice that $\mathbf{Mod}(\mathscr{M}_2(O_2)/\mathcal{I}(O_2)) = \mathbf{Mod}(\mathscr{M}_2(O_2) : \mathscr{R}_2(O_1^2 \hookrightarrow O_2^2))$ in the case of $|\mathcal{I}(O_2)| = 1$. We get the isomorphism as desired. □

Corollary 3.5.1 *Let* $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ *be an m-module with* $O = \{ +_i \mid 1 \leq i \leq m\}$, $O_1 = \{\cdot_i | 1 \leq i \leq m\}$, $O_2 = \{\dot{+}_i | 1 \leq i \leq m\}$, *$M$ a module on a ring* $(R; +, \cdot)$ *and* $\iota : \mathbf{Mod}(\mathscr{M}_1(O_1) : \mathscr{R}_1(O_1^1 \hookrightarrow O_2^1)) \to M$ *a onto homomorphism with* $\mathrm{Ker}\iota = \mathscr{N}$. *Then there exists a representation pair* $(R', \widetilde{P})$ *such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R',\widetilde{P})} \simeq M,$$

particularly, if $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ *is a module* $\mathscr{M}$, *then*

$$\mathscr{M}/\mathscr{N} \simeq M.$$

3.5.2 Finite Dimensional Multi-Module. For constructing multi-submodules of an m-module $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ with $O = \{ +_i \mid 1 \leq i \leq m\}$, $O_1 = \{\cdot_i | 1 \leq i \leq m\}$, $O_2 = \{\dot{+}_i | 1 \leq i \leq m\}$, a general way is described in the following.

Let $\widehat{S} \subset \mathscr{M}$ with $|\widehat{S}| = n$. Define its *linearly spanning set* $\left\langle \widehat{S} | \mathscr{R} \right\rangle$ in $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ to be

$$\left\langle \widehat{S} | \mathscr{R} \right\rangle = \left\{ \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \alpha_{ij} \times_i x_{ij} | \ \alpha_{ij} \in \mathscr{R}, x_{ij} \in \widehat{S} \right\},$$

where

$$\begin{aligned} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} \times_{ij} x_i \ = \ & a_{11} \times_1 x_{11} +_1 \cdots +_1 a_{1n} \times_1 x_{1n} \\ & +^{(1)} a_{21} \times_2 x_{21} +_2 \cdots +_2 a_{2n} \times_2 x_{2n} \\ & +^{(2)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +^{(3)} \\ & a_{m1} \times_m x_{m1} +_m \cdots +_m a_{mn} \times_m x_{mn} \end{aligned}$$

with $+^{(1)}, +^{(2)}, +^{(3)} \in O$ and particularly, if $+_1 = +_2 = \cdots = +_m$, it is denoted by $\sum\limits_{i=1}^{m} x_i$ as usual. It can be checked easily that $\left\langle \widehat{S} | \mathscr{R} \right\rangle$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(O)$:

$\mathscr{R}(O_1 \hookrightarrow O_2)$), call it *generated by* $\widehat{S}$ *in* $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$. If $\widehat{S}$ is finite, we also say that $\left\langle \widehat{S}|\mathscr{R} \right\rangle$ is *finitely generated*. Particularly, if $\widehat{S} = \{x\}$, then $\left\langle \widehat{S}|\mathscr{R} \right\rangle$ is called a *cyclic multi-submodule of* $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$, denoted by $\mathscr{R}x$. Notice that

$$\mathscr{R}x = \left\{ \bigoplus_{i=1}^{m} a_i \times_i x \mid a_i \in \mathscr{R} \right\}$$

by definition. For any finite set $\widehat{S}$, if for any integer s, $1 \le s \le m$,

$$\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{s_i} \alpha_{ij} \times_i x_{ij} = 0_{+_s}$$

implies that $\alpha_{ij} = 0_{\dot{+}_s}$ for $1 \le i \le m, 1 \le j \le n$, then we say that $\{x_{ij} | 1 \le i \le m, 1 \le j \le n\}$ is independent and $\widehat{S}$ a *basis of the multi-module* $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$, denoted by $\left\langle \widehat{S}|\mathscr{R} \right\rangle = \mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$.

For a multi-ring $(\mathscr{R}; O_1 \hookrightarrow O_2)$ with a unit $1.$ for $\forall \cdot \in O_1$, where $O_1 = \{\cdot_i | 1 \le i \le m\}$ and $O_2 = \{\dot{+}_i | 1 \le i \le m\}$, let

$$\mathscr{R}^{(n)} = \{(x_1, x_2, \cdots, x_n) | \ x_i \in \mathscr{R}, 1 \le i \le n\}.$$

Define operations

$$(x_1, x_2, \cdots, x_n) +_i (y_1, y_2, \cdots, y_n) = (x_1 \dot{+}_i y_1, x_2 \dot{+}_i y_2, \cdots, x_n \dot{+}_i y_n),$$

$$a \times_i (x_1, x_2, \cdots, x_n) = (a \cdot_i x_1, a \cdot_i x_2, \cdots, a \cdot_i x_n)$$

for $\forall a \in \mathscr{R}$ and integers $1 \le i \le m$. Then it can be immediately known that $\mathscr{R}^{(n)}$ is a multi-module $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(O_1 \hookrightarrow O_2))$. We construct a basis of this special multi-module in the following.

For any integer k, $1 \le k \le n$, let

$$\mathbf{e}_1 = (1_{\cdot_k}, 0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k});$$

$$\mathbf{e}_2 = (0_{\dot{+}_k}, 1_{\cdot_k}, \cdots, 0_{\dot{+}_k});$$

$$\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$\mathbf{e}_n = (0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k}, 1_{\cdot_k}).$$

Notice that

$$(x_1, x_2, \cdots, x_n) = x_1 \times_k \mathbf{e}_1 +_k x_2 \times_k \mathbf{e}_2 +_k \cdots +_k x_n \times_k \mathbf{e}_n.$$

We find that each element in $\mathscr{R}^{(n)}$ is generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Now since

$$(x_1, x_2, \cdots, x_n) = (0_{+_k}, 0_{+_k}, \cdots, 0_{+_k})$$

implies that $x_i = 0_{+_k}$ for any integer $i, 1 \leq i \leq n$. Whence, $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is a basis of $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(O_1 \hookrightarrow O_2))$.

Theorem 3.5.2 *Let* $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2)) = \left\langle \widehat{S} | \mathscr{R} \right\rangle$ *be a finitely generated multi-module with* $\widehat{S} = \{u_1, u_2, \cdots, u_n\}$. *Then*

$$\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2)) \simeq \mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(O_1 \hookrightarrow O_2)).$$

Proof Define a mapping $\vartheta : \mathscr{M}(O) \to \mathscr{R}^{(n)}$ by $\vartheta(u_i) = \mathbf{e}_i$, $\vartheta(a \times_j u_i) = a \times_j \mathbf{e}_j$ and $\vartheta(u_i +_k u_j) = \mathbf{e}_i +_k \mathbf{e}_j$ for any integers i, j, k, where $1 \leq i, j, k \leq n$. Then we know that

$$\vartheta(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} \times_i u_i) = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} \times_i \mathbf{e}_i.$$

Whence, ϑ is a homomorphism. Notice that it is also $1-1$ and onto. We know that ϑ is an isomorphism between $\mathbf{Mod}(\mathscr{M}(O) : \mathscr{R}(O_1 \hookrightarrow O_2))$ and $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(O_1 \hookrightarrow O_2))$. □

§3.6 RESEARCH PROBLEMS

3.6.1 The conceptions of bi-group and bi-subgroup were first appeared in [Mag1] and [MaK1]. Certainly, they are special cases of multi-group and multi-subgroup. More results on bi-groups can be found in [Kan1]. We list some open problems in the following.

Problem 3.6.1 *Establish a decomposition theory for multi-groups.*

In group theory, we know the following decomposition result for groups.

Theorem 3.6.1([Rob1]) *Let G be a finite Ω-group. Then G can be uniquely decomposed as a direct product of finite non-decomposition Ω-subgroups.*

Theorem 3.6.2([Wan1]) *Each finite Abelian group is a direct product of its Sylow p-subgroups.*

Then Problem 3.6.1 can be restated as follows.

Problem 3.6.2 *Whether can we establish a decomposition theory for multi-groups similar to the above two results in group theory, especially, for finite multi-groups?*

Problem 3.6.2 *Define the conception of simple multi-groups. For finite multi-groups, whether can we find all simple multi-groups?*

We have known that there are four simple group classes following ([XHLL1]):

Class 1: The cyclic groups of prime order;

Class 2: The alternating groups $A_n, n \geq 5$;

Class 3: The 16 groups of Lie types;

Class 4: The 26 sporadic simple groups.

Problem 3.6.3 *Determine the structure properties of multi-groups generated by finite elements.*

For a subset A of a multi-group $\widetilde{G}$, define its spanning set by

$$\langle A\rangle = \left\{a \circ b | a, b \in A \text{ and } \circ \in O(\widetilde{G})\right\}.$$

If there exists a subset $A \subset \widetilde{G}$ such that $\widetilde{G} = \langle A\rangle$, then call $\widetilde{G}$ is generated by A. Call $\widetilde{G}$ *finitely generated* if there exist a finite set A such that $\widetilde{G} = \langle A\rangle$. Then Problem 3.6.3 can be restated as follows:

Problem 3.6.4 *Can we establish a finite generated multi-group theory similar to that of finite generated groups?*

Problem 3.6.5 *Determine the structure of a Noether multi-ring.*

3.6.2 A ring R is called to be a *Noether ring* if its every ideal chain only has finite terms. Similarly, for a multi-ring $\widetilde{R}$, if its every multi-ideal chain only has finite terms, it is called to be a *Noether multi-ring*.

Problem 3.6.6 *Can we find the structures of Noether multi-rings likewise that of Corollary* 3.3.3 *and Theorem* 3.3.6*?*

Problem 3.6.7 *Define a Jacobson or Brown-McCoy radical for multi-rings similar to that in rings, and determine their contribution to multi-rings.*

3.6.3 Notice that Theorems 3.4.2 and 3.4.3 imply that we can establish a linear theory for multi-vector spaces, but the situation is complex than that of classical linear spaces. The following problems are interesting.

Problem 3.6.8 *Similar to that of linear spaces, define linear transformations on vector multi-spaces. Can we establish a matrix theory for those linear transformations?*

Problem 3.6.9 *Whether a vector multi-space must be a linear space?*

Conjecture 3.6.1 *There exists non-linear vector multi-spaces in vector multi-spaces.*

If Conjecture 3.6.1 is true, there is a fundamental problem on vector multi-spaces should be considered following.

Problem 3.6.10 *Can we apply vector multi-spaces to those non-linear spaces?*

3.6.4 For a complete multi-space $\left(\widetilde{A}; O\left(\widetilde{A}\right)\right)$, we can get a *multi-operation system* $\widetilde{A}$. For example, if $\widetilde{A}$ is a multi-field $\widetilde{F} = \bigcup\limits_{i=1}^{n} F_i$ with a double operation set $O\left(\widetilde{F}\right) = \{(+_i, \times_i) | \ 1 \le i \le n\}$, then $\left(\widetilde{F}; +_1, +_2, \cdots, +_n\right)$, $\left(\widetilde{F}; \times_1, \times_2, \cdots, \times_n\right)$ and $\left(\widetilde{F}; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n)\right)$ are multi-operation systems. By this view, the classical operation system $(R\ ; +)$ and $(R\ ; \times)$ are systems with one operation. For a multi-operation system $\widetilde{A}$, we can define these conceptions of equality and inequality, $\cdots$, etc.. For example, in the multi-operation system $\left(\widetilde{F}; +_1, +_2, \cdots, +_n\right)$, we define the equalities $=_1, =_2, \cdots, =_n$ such as those in sole operation systems $\left(\widetilde{F}; +_1\right), \left(\widetilde{F}; +_2\right), \cdots, \left(\widetilde{F}; +_n\right)$, for example, $2 \ =_1 \ 2, 1.4 \ =_2 \ 1.4, \cdots, \sqrt{3} =_n \sqrt{3}$ which is the same as the usual meaning and similarly, for the conceptions $\ge_1, \ge_2, \cdots, \ge_n$ and $\le_1, \le_2, \cdots, \le_n$.

In the classical operation system $(R; +)$, the equation system

$$\begin{aligned} x + 2 + 4 + 6 &= 15 \\ x + 1 + 3 + 6 &= 12 \\ x + 1 + 4 + 7 &= 13 \end{aligned}$$

can not has a solution. But in $\left(\widetilde{F}; +_1, +_2, \cdots, +_n\right)$, the equation system

$$\begin{aligned} x +_1 2 +_1 4 +_1 6 &=_1 15 \\ x +_2 1 +_2 3 +_2 6 &=_2 12 \\ x +_3 1 +_3 4 +_3 7 &=_3 13 \end{aligned}$$

has a solution x if

$$\begin{aligned} 15 +_1 (-1) +_1 (-4) +_1 (-16) &= 12 +_2 (-1) +_2 (-3) +_2 (-6) \\ &= 13 +_3 (-1) +_3 (-4) +_3 (-7). \end{aligned}$$

in $(\widetilde{F}; +_1, +_2, \cdots, +_n)$. Whence, an element maybe have different disguises in a multi-operation system.

Problem 3.6.11 *Find necessary and sufficient conditions for a multi-operation system with more than* 3 *operations to be the rational number field Q, the real number field R or the complex number field C.*

For a multi-operation system $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$ and integers $a, b, c \in N$, if $a = b \times_i c$ for an integer i, $1 \leq i \leq n$, then b and c are called *factors* of a. An integer p is called a *prime* if there exist integers n_1, n_2 and i, $1 \leq i \leq n$ such that $p = n_1 \times_i n_2$, then $p = n_1$ or $p = n_2$. Two problems for primes of a multi-operation system $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$ are presented in the following.

Problem 3.6.12 *For a positive real number x, denote by $\pi_m(x)$ the number of primes $\leq x$ in $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$. Determine or estimate $\pi_m(x)$.*

Notice that for the positive integer system, by a well-known theorem, i.e., *Gauss prime theorem*, we have known that

$$\pi(x) \sim \frac{x}{\log x}.$$

Problem 3.6.13 *Find the additive number properties for $(N\ ; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$, for example, we have weakly forms for Goldbach's conjecture and Fermat's problem as follows.*

Conjecture 3.6.2 *For any even integer $n, n \geq 4$, there exist odd primes p_1, p_2 and an integer $i, 1 \leq i \leq n$ such that $n = p_1 +_i p_2$.*

Conjecture 3.6.3 *For any positive integer q, the Diophantine equation $x^q + y^q = z^q$ has non-trivial integer solutions (x, y, z) at least for an operation "$+_i$" with $1 \leq i \leq n$.*

3.6.5 A *Smarandache n-structure on a set S* means a weak structure $\{w(0)\}$ on S such that there exists a chain of proper subsets $P(n-1) \subset P(n-2) \subset \cdots \subset P(1) \subset S$ whose corresponding structures verify the inverse chain $\{w(n-1)\} \supset \{w(n-2)\} \supset \cdots \supset \{w(1)\} \supset \{w(0)\}$, i.e., structures satisfying more axioms.

Problem 3.6.14 *For Smarandache multi-structures, solves Problems* 3.6.1 – 3.6.10.

CHAPTER 4.

Multi-Voltage Graphs

There is a convenient way for constructing regular covering spaces of a graph G in topological graph theory, i.e., by a voltage assignment $\alpha : G \to \Gamma$ on G, first introduced by Gustin in 1963 and then generalized by Gross in 1974, where $(\Gamma; \circ)$ is a finite group. Youngs extensively used voltage graphs in proving Heawood map coloring theorem. Today, this approach has been also applied for finding regular maps on surface. However, there are few attentions on irregular coverings of graphs. We generalize such graphs G by a voltage assignment $\alpha : G \to \Gamma$ to $\alpha : G \to \widetilde{\Gamma}$, i.e., multi-voltage graphs, where $(\widetilde{\Gamma}; O)$ is a finite multi-group. By applying results in last chapter, two kind of multi-voltage graphs are introduced for finding irregular coverings of graphs. Elementary properties and results on these multi-voltage graphs are obtained in Sections 4.2-4.3. Furthermore, we also construct graph models for algebraic multi-systems, including Cayley graphs on multi-groups in Section 4.4 and get results on structural properties of algebraic systems by such graph models.

§4.1 VOLTAGE GRAPHS

4.1.1 Voltage Graph. Let G be a connected graph and $(\Gamma;\circ)$ a group. For each edge $e\in E(G), e=uv$, an *orientation* on e is such an orientation on e from u to v, denoted by $e=(u,v)$, called the *plus orientation* and its *minus orientation*, from v to u, denoted by $e^{-1}=(v,u)$. For a given graph G with plus and minus orientation on edges, a *voltage assignment* on G is a mapping σ from the plus-edges of G into a group Γ satisfying $\sigma(e^{-1})=\sigma^{-1}(e), e\in E(G)$. These elements $\sigma(e), e\in E(G)$ are called voltages, and (G,σ) a *voltage graph* with a voltage assignment $\sigma: G\to\Gamma$.

For a voltage graph (G,σ) with a voltage assignment $\sigma: G\to\Gamma$, its lifting $G^\sigma=(V(G^\sigma),E(G^\sigma);I(G^\sigma))$ is defined by

$$V(G^\sigma)=V(G)\times\Gamma,\ \text{ and } \forall(u,a)\in V(G)\times\Gamma \text{ is abbreviated to } u_a,$$
$$E(G^\sigma)=\{(u_a,v_{a\circ b})|e^+=(u,v)\in E(G),\sigma(e^+)=b\}$$

and

$$I(G^\sigma)=\{(u_a,v_{a\circ b})|I(e)=(u_a,v_{a\circ b})\ if\ e=(u_a,v_{a\circ b})\in E(G^\sigma)\}.$$

For example, let $G=K_3$ and $\Gamma=Z_2$. Then the voltage graph (K_3,σ) with $\sigma: K_3\to Z_2$ and its lifting are shown in Fig.4.1.1.

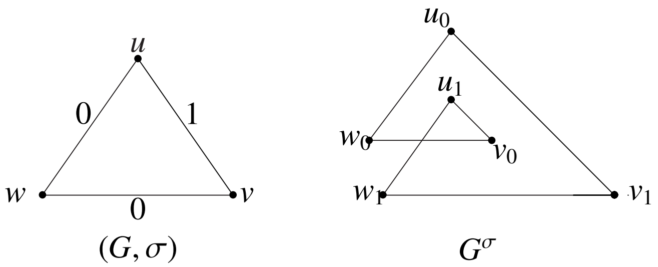


Fig.6.1.1

Let $(G;\sigma)$ be a voltage graph with a voltage assignment $\sigma: G\to\Gamma$. Then for $\forall v\in V(G)$ and $e\in E(G)$, the sets

$$[v]^\Gamma=\{\,v_a\mid a\in\Gamma\,\},\quad [e]^\Gamma=\{\,e_a\mid a\in\Gamma\,\}$$

are defined the fibers over v or e, respectively and $p: G^\sigma\to G$ determined by $p: v_a\to v$ and $e_a\to e$ for $v\in V(G)$, $e\in E(G)$ and $a\in\Gamma$ the *natural projection* of $(G;\sigma)$. Clearly, p is a $|G|$-sheet covering for any point $x\in G$.

4.1.2 Lifted Walk. For a walk $W = e_1^{\sigma_1},\ e_2^{\sigma_2},\ \cdots,\ e_n^{\sigma_n}$ with $\sigma_i \in \{+,-\}$, define its *net voltage* by

$$\sigma(W) = \sigma(e_1)\sigma(e_2)\cdots\sigma(e_n).$$

For example, the net voltage on the walk uv^+, vw^+, wv^- in Fig.4.1.1 is $1 + 0 + 0 = 1$. A *lifting* of such a walk W is determined by $\widetilde{W} = \widetilde{e}_1^{\sigma_1},\ \widetilde{e}_2^{\sigma_2},\ \cdots,\ \widetilde{e}_n^{\sigma_n}$ such that $p(\widetilde{e}_i) \in [e_i]^\Gamma$ for integers $1 \leq i \leq n$. For instance, the liftings of the walk uv^+, vw^+, wv^- in Fig.4.1.1 are $u_0v_1^+, v_1w_1^+, w_1v_1^-$ and $u_1v_0^+, v_0w_0^+, w_0v_0^-$. Particularly, let $e^+ = (u, v) \in E(G)$ with $\sigma(e^+) = b$, $o(b) = n$, we get an n-circuit starting at u_a, i.e.,

$$u_a, e_a^+, u_{a\circ b}, e_{a\circ b}^+, u_{a\circ b^2}, e_{a\circ b^2}^+, \cdots, u_{a\circ b^{n-1}}, e_{a\circ b^{n-1}}^+, u_{a\circ b^n} = u_a$$

in the lifting G^σ.

Theorem 4.1.1 *If W is a walk in a voltage graph $(G;\sigma)$ with a voltage assignment $\sigma : G \rightarrow \Gamma$ such that the initial vertex of W is u, then for each vertex u_a in the fiber $[u]$ there is a unique lifting of W that starts at u_a.*

Proof Assume $W = u, e_1^{\sigma_1}, v_1, e_2^{\sigma_2}, v_2, \cdots$. If $\sigma_1 = +$, then, since there is only one plus-directed edge, i.e., the edge e_1^+ in the fiber $[e_1]^\Gamma$ starts at vertex u_a, the edge must be the first edge in the lifting of W starting at u_a. If $\sigma = -$, similarly, since there is only one minus-directed edge, i.e., $(e_1)^-_{a\circ\sigma(e^-)}$ in the fiber $[e_1]^\Gamma$ starts at u_a, it follows the edge must be the first edge in the lifting of W starting at u_a. Similarly, there is only one possible choice of the second edge $e_2^{\sigma_2}$ in the lifting of W because its initial vertex must be the terminal vertex of the first edge and its lifting must in the fiber $[e_2]^\Gamma$. Continuing this process, the uniqueness of lifting walk W starting at u_a holds. □

Theorem 4.1.2 *If W is a walk from u to v in a voltage graph $(G;\sigma)$ with a voltage assignment $\sigma : G \rightarrow \Gamma$ and $\sigma(W) = b$, then the lifting W_a starting at u_a terminates at the vertex $v_{a\circ b}$.*

Proof Let $b_1, b_2, \cdots, b_l$ be the successive voltage encountered on a traversal of walk W. Then it is clear that these subscripts of the successive vertices on the lifting W^σ of W_a are $a, s \circ b_1, a \circ b_1 \circ b_2, \cdots, a \circ b_1 \circ b_2 \circ \cdots \circ b_l = a \circ b$. Thus the terminal vertex of W^σ starts at u_a is $v_{a\circ b}$. □

Corollary 4.1.1 *Let $P(u, v)$ be a path from u to v in a voltage graph $(G;\sigma)$ with a voltage assignment $\sigma : G \rightarrow \Gamma$ and $\sigma(P(u, v)) = b$. then the lifting of $P(u, v)$ is a path $P(u_a, v_{a\circ b})$.*

Furthermore, if W is a circuit in $(G;\sigma)$, we get the following result.

Theorem 4.1.3 *Let C be a circuit of length m in a voltage graph $(G;\sigma)$ with a voltage assignment $\sigma : G \to \Gamma$ and $o(\sigma(C)) = n$. Then each connected component of $p^{-1}(C)$ is a circuit of length mn, and there are $\frac{|\Gamma|}{n}$ such components.*

Proof Let C be the walk $W = u, e_1^{\sigma_1}, v, e_2^{\sigma_2}, \cdots, e_m^{\sigma_m}, u$, $\sigma_i \in \{+,-\}$, $\sigma(W) = b$ and $u_a \in [u]^\Gamma$. Applying Theorem 4.1.2, we know that the component of $p^{-1}(C)$ containing u_a is formed by edges in walks

$$W_a, W_{a\circ b}, \cdots, W_{a\circ b^{n-1}},$$

which form a circuit of length mn by edges in these walk attached end-by-end. Notice that there are $\frac{|\Gamma|}{\langle b\rangle}$ left cosets of the cyclic group $\langle b\rangle$ in $(\Gamma;\circ)$ and each of them uniquely determine a component of p^{-1}. Thus there are $\frac{|\Gamma|}{n}$ lifted circuits of length C. $\square$

4.1.3 Group Action. Let G be a graph and $(\Gamma;\circ)$ a group. If for each element $g \in \Gamma$, there is an automorphism ϕ_g of G such that the following two conditions hold:

(1) ϕ_{1_Γ} is the identity automorphism of G;

(2) $\phi_g \cdot \phi_h = \phi_{g\circ h}$ for $g, h \in \Gamma$,

then the group $(\Gamma;\circ)$ is said to *act on the graph* G. For $\forall v \in V(G)$, $e \in E(G)$, the sets

$$v^\Gamma = \{\, v^g \mid g \in \Gamma \,\} \quad \text{and} \quad e^\Gamma = \{\, e^g \mid g \in \Gamma \,\}$$

are called the vertex orbit or edge orbit under the action of $(\Gamma;\circ)$. The sets of vertex orbits and edge orbits are respectively denoted by V/Γ and E/Γ. Moreover, if the additional condition

(3) For each element $1_\Gamma \neq g \in \Gamma$, there are no vertex $v \in V(G)$ such that $\phi_g(v) = v$ and no edge $e \in E(G)$ such that $\phi_g(e) = e$

holds, then $(\Gamma;\circ)$ is said to *act freely* on G.

The *regular quotient* G/Γ is such a graph with vertex set V/Γ and edge set E/Γ such that a vertex orbit v^Γ is an end-vertex of the orbit e^Γ if any vertex v in v^Γ is an end-vertex of an edge in e^Γ. There are easily to verify that such a graph G/Γ is well-defined, i.e., e is an edge with an end-vertex v if and only if e^Γ with an end-vertex v^Γ.

Now let $(G;\sigma)$ be a voltage graph with a voltage assignment $\sigma : G \to \Gamma$. There is a natural action of $(\Gamma,\circ)$ on G^σ by rules $\phi_g(v_a) = v_{v_{g\circ a}}$ on vertices and $\phi_g(e_a) = e_{g\circ a}$ on edges for $g \in \Gamma$. Such an action ϕ_g is an automorphism of G^Γ by verifying that $\phi_g \cdot \phi_h = \phi_{g\circ h}$. Then the following result is clear by definition.

Theorem 4.1.4 *Let* $(G;\sigma)$ *be a voltage graph with a voltage assignment* $\sigma : G \to \Gamma$ *and* $v_a \in V(G^\sigma)$, $e_a \in E(G^\sigma)$. *Then* $v_a^\Gamma = p^{-1}(v)$ *and* $e_a^\Gamma = p^{-1}(e)$.

Proof For $v_a \in V(G^\sigma)$, by definition we know that

$$v_a^\Gamma = \{\ \phi_g(v_a) = v_{g\circ a} \mid g \in \Gamma\ \} = \{\ v_h \mid h \in \Gamma\} = p^{-1}(v).$$

Similarly, we get $e_a^\Gamma = p^{-1}(e)$. □

4.1.4 Lifted Graph. For a voltage graph $(G;\sigma)$ with a voltage assignment $\sigma : G \to \Gamma$, we know that $(\Gamma;\circ)$ is act-free on G^σ because if $\phi_g(v_a) = v_a$ or $\phi_g(e_a) = e_a$, then $g = 1_\Gamma$. This fact enables Gross and Tucker found a necessary and sufficient condition for a graph being that lifting of a voltage graph following.

Theorem 4.1.5(Gross and Tucker, 1974) *Let* $(\Gamma;\circ)$ *be a group acting freely on a graph* $\widetilde{G}$ *and* $G = \widetilde{G}/\Gamma$. *Then there is a voltage assignment* $\sigma : G \to \Gamma$ *and a labeling of vertices on* G *by elements of* $V(G) \times \Gamma$ *such that* $\widetilde{G} = G^\sigma$ *and the action is the natural action of* $(\Gamma;\circ)$ *on* G^σ.

Proof First, we choose positive directions for edges in the graph G and $\widetilde{G}$ so that the quotient map $q_\Gamma : \widetilde{G} \to G$ is direction-preserving and that the action of $(\Gamma;\circ)$ on $\widetilde{G}$ preserves directions. Second, for $\forall v \in V(G)$, label one vertex of the orbit $p^{-1}(v)$ in $\widetilde{G}$ as v_{1_Γ} and for every element $g \in \Gamma$, $g \neq 1_\Gamma$, label the vertex $\phi_a(v_1^\Gamma)$ as v_a. Now if the edge e of G runs from u to w, we assigns the label e_a to the edge of the orbit $p^{-1}(e)$ that originates at the vertex v_a. Since $(\Gamma;\circ)$ acts freely on $\widetilde{G}$, there are just $|\Gamma|$ edges in the orbit $p^{-1}(e)$, one originating at each of the vertices in the vertex orbit $p^{-1}(v)$. Thus, the choice of an edge to be labelled e_a is unique. Finally, if the terminal vertex of the edge e_{1_Γ} is w_b, one assigns a voltage b to the edge e in graph G. Thus $\sigma(e^+) = b$. To show that this labelling of edges in $p^{-1}(e)$ and the choice of voltages b for the edge e really yields an isomorphism $\vartheta : \widetilde{G} \to G^\sigma$, one needs to show that for $\forall a \in \Gamma$ that the edge e_a terminates at the vertex $w_{a\circ b}$. However, since $e_a = \phi_a(e_{1_\Gamma})$, the terminal vertex of the edge e_a must be the terminal vertex of the edge $\phi_a(e_{1_{\Gamma_i}})$, i.e.,

$$\phi_a(w_b) = \phi_a \cdot \phi_b(w_{1_\Gamma}) = \phi_(a \circ b)(w_{1_\Gamma}) = w_{a\circ b}.$$

Under this labelling process, the isomorphism $\vartheta : \widetilde{G} \to G^{\sigma}$ identifies orbits in $\widetilde{G}$ with fibers of G^{σ}. Moreover, it is defined precisely so that the action of $(\Gamma; \circ)$ on $\widetilde{G}$ is consistent with the natural action on the lifted graph G^{σ}. This completes the proof. $\square$

§4.2 MULTI-VOLTAGE GRAPHS–TYPE I

4.2.1 Multi-Voltage Graph of Type I. The first type of multi-voltage graph is labeling edges in a graph by elements in a finite multi-group $\left(\widetilde{\Gamma}; O\right)$. Formally, it is defined in the following.

Definition 4.2.1 *Let* $\left(\widetilde{\Gamma}; O\right)$ *be a finite multi-group with* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$, $O\left(\widetilde{\Gamma}\right) = \{\circ_i | 1 \leq i \leq n\}$ *and* G *a graph. If there is a mapping* $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma}$ *such that* $\psi(e^{-1}) = (\psi(e^{+}))^{-1}$ *for* $\forall e^{+} \in X_{\frac{1}{2}}(G)$*, then the 2-tuple* (G, ψ) *is called a multi-voltage graph of type I.*

Geometrically, a multi-voltage graph is nothing but a weighted graph with weights in a multi-group. Similar to voltage graphs, the importance of a multi-voltage graph is in its *lifting* defined in the definition following.

Definition 4.2.2 *For a multi-voltage graph* (G, ψ) *of type I, its lifting graph* $G^{\psi} = (V(G^{\psi}), E(G^{\psi}); I(G^{\psi}))$ *is defined by*

$$V(G^{\psi}) = V(G) \times \widetilde{\Gamma},$$

$$E(G^{\psi}) = \{(u_a, v_{a\circ b}) | e^{+} = (u, v) \in X_{\frac{1}{2}}(G), \psi(e^{+}) = b, a \circ b \in \widetilde{\Gamma}\}$$

and

$$I(G^{\psi}) = \{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^{\psi})\}.$$

For abbreviation, a vertex (x, g) in G^{ψ} is also denoted by x_g. Now for $\forall v \in V(G)$, $v \times \widetilde{\Gamma} = \{v_g | g \in \widetilde{\Gamma}\}$ is called a *fiber over* v, denoted by F_v. Similarly, for $\forall e^{+} = (u, v) \in X_{\frac{1}{2}}(G)$ with $\psi(e^{+}) = b$, all edges $\{(u_g, v_{g\circ b}) | g, g \circ b \in \widetilde{\Gamma}\}$ is called the *fiber over* e, denoted by F_e.

For a multi-voltage graph (G, ψ) and its lifting G^{ψ}, there is also a *natural projection* $p : G^{\psi} \to G$ defined by $p(F_v) = v$ for $\forall v \in V(G)$. It can be verified easily that $p(F_e) = e$ for $\forall e \in E(G)$.

Foe example, choose $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2$ with $\Gamma_1 = \{1, a, a^2\}$, $\Gamma_2 = \{1, b, b^2\}$ and $a \neq b$. A multi-voltage graph and its lifting are shown in Fig.4.2.1.

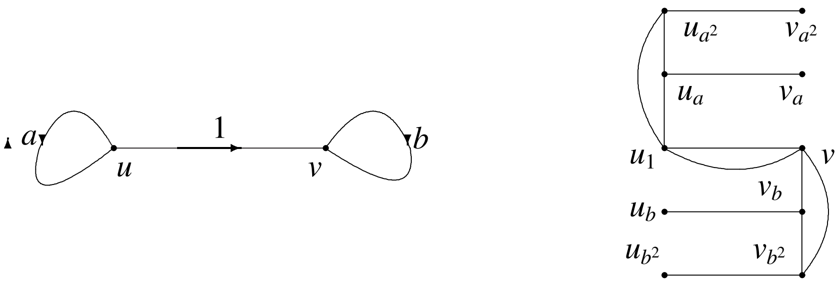


Fig 4.2.1

Let $\left(\widetilde{\Gamma}; O\right)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$, $O = \{\circ;\ 1 \leq i \leq n\}$. We know the liftings of walks in multi-voltage graphs of type I similar to that of voltage graphs following.

Theorem 4.2.1 *Let $W = e^1 e^2 \cdots e^k$ be a walk in a multi-voltage graph (G, ψ) with initial vertex u. Then there exists a lifting W^{ψ} start at u_a in G^{ψ} if and only if there are integers $i_1, i_2, \cdots, i_k$ such that*

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}} \text{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$

Proof Consider the first semi-arc in the walk W, i.e., e_1^+. Each lifting of e_1 must be $(u_a, u_{a\circ\psi(e_1^+)})$. Whence, there is a lifting of e_1 in G^{ψ} if and only if there exists an integer i_1 such that $\circ = \circ_{i_1}$ and $a, a \circ_{i_1} \psi(e_1^+) \in \Gamma_{i_1}$.

Now if we have proved there is a lifting of a sub-walk $W_l = e_1 e_2 \cdots e_l$ in G^{ψ} if and only if there are integers $i_1, i_2, \cdots, i_l$, $1 \leq l < k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \quad \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq l$, we consider the semi-arc e_{l+1}^+. By definition, there is a lifting of e_{l+1}^+ in G^{ψ} with initial vertex $u_{a\circ_{i_1}\psi(e_1^+)\circ_{i_2}\cdots\circ_{i_{j-1}}\psi(e_l^+)}$ if and only if there exists an integer i_{l+1} such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_l^+) \in \Gamma_{l+1} \text{ and } \psi(e_{l+1}^+) \in \Gamma_{l+1}.$$

Whence, by the induction principle, there exists a lifting W^{ψ} start at u_a in G^{ψ} if and only if there are integers $i_1, i_2, \cdots, i_k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \text{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$. □

For two elements $g, h \in \widetilde{\Gamma}$, if there exist integers $i_1, i_2, \cdots, i_k$ such that $g, h \in \bigcap\limits_{j=1}^{k} \Gamma_{i_j}$ but for $\forall i_{k+1} \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_k\}$, $g, h \notin \bigcap\limits_{j=1}^{k+1} \Gamma_{i_j}$, we call $k = \Pi[g, h]$ the *joint number of g and h*. Denote by $O(g, h) = \{\circ_{i_j}; 1 \leq j \leq k\}$ and define $\widetilde{\Pi}[g, h] = \sum\limits_{\circ \in O(\widetilde{\Gamma})} \Pi[g, g \circ h]$, where $\Pi[g, g \circ h] = \Pi[g \circ h, h] = 0$ if $g \circ h$ does not exist in $\widetilde{\Gamma}$. According to Theorem 4.2.1, we get an upper bound for the number of liftings in G^{ψ} for a walk W in (G, ψ) following.

Corollary 4.2.1 *If those conditions in Theorem* 4.2.1 *hold, the number of liftings of W with initial vertex u_a in G^{ψ} is not excess*

$$\widetilde{\Pi}\left[a, \psi(e_1^+)\right] \times$$
$$\prod_{i=1}^{k} \sum_{\circ_1 \in O(a, \psi(e_1^+))} \cdots \sum_{\circ_i \in O(a; \circ_j, \psi(e_j^+), 1 \leq j \leq i-1)} \widetilde{\Pi}\left[a \circ_1 \psi(e_1^+) \circ_2 \cdots \circ_i \psi(e_i^+), \psi(e_{i+1}^+)\right],$$

where $O(a; \circ_j, \psi(e_j^+), 1 \leq j \leq i-1) = O(a \circ_1 \psi(e_1^+) \circ_2 \cdots \circ_{i-1} \psi(e_{i-1}^+), \psi(e_i^+))$.

The natural projection of a multi-voltage graph is not regular in general. For finding a regular covering of a graph, a typical class of multi-voltage graphs is the case of $\Gamma_i = \Gamma$ for any integer $i, 1 \leq i \leq n$ in these multi-groups $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$. In this case, we can find the exact number of liftings in G^{ψ} for a walk in (G, ψ) following.

Theorem 4.2.2 *Let $\left(\widetilde{\Gamma}; O\right)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ and $O = \{\circ_i; 1 \leq i \leq n\}$ and let $W = e^1 e^2 \cdots e^k$ be a walk in a multi-voltage graph (G, ψ), $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma}$ of type I with initial vertex u. Then there are n^k liftings of W in G^{ψ} with initial vertex u_a for $\forall a \in \widetilde{\Gamma}$.*

Proof The existence of lifting of W in G^{ψ} is obvious by Theorem 4.2.1. Consider the semi-arc e_1^+. Since $\Gamma_i = \Gamma$ for $1 \leq i \leq n$, we know that there are n liftings of e_1 in G^{ψ} with initial vertex u_a for any $a \in \widetilde{\Gamma}$, each with a form $(u_a, u_{a \circ \psi(e_1^+)})$, $\circ \in O\left(\widetilde{\Gamma}\right)$.

Now if we have gotten $n^s, 1 \leq s \leq k-1$ liftings in G^{ψ} for a sub-walk $W_s = e^1 e^2 \cdots e^s$. Consider the semi-arc e_{s+1}^+. By definition we know that there are also n liftings of e_{s+1} in G^{ψ} with initial vertex $u_{a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_s \psi(e_s^+)}$, where $\circ_i \in O\left(\widetilde{\Gamma}\right), 1 \leq i \leq s$. Whence, there are n^{s+1} liftings in G^{ψ} for a sub-walk $W_s = e^1 e^2 \cdots e^{s+1}$ in $(G; \psi)$.

By the induction principle, we know the assertion is true. □

Particularly, if $\left(\widetilde{\Gamma}; O\right)$ is nothing but a group, i.e., $\circ_i = \circ$ for integers $1 \leq i \leq n$, we get Theorem 4.1.1 again.

Corollary 4.2.2 *Let W be a walk in a voltage graph $(G,\psi), \psi : X_{\frac{1}{2}}(G) \rightarrow \Gamma$ with initial vertex u. Then there is an unique lifting of W in G^{ψ} with initial vertex u_a for $\forall a \in \Gamma$.*

If a lifting W^{ψ} of a multi-voltage graph (G,ψ) is the same as the lifting of a voltage graph $(G,\alpha), \alpha : X_{\frac{1}{2}}(G) \rightarrow \Gamma_i$, then this lifting is a *homogeneous lifting of* Γ_i. For lifting a circuit in a multi-voltage graph, we get the following result.

Theorem 4.2.3 *Let $\left(\widetilde{\Gamma}; O\right)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ and $O = \{\circ_i; 1 \leq i \leq n\}$, $C = u_1u_2 \cdots u_m u_1$ a circuit in a multi-voltage graph (G,ψ) and $\psi : X_{\frac{1}{2}}(G) \rightarrow \widetilde{\Gamma}$. Then there are $\dfrac{|\Gamma|}{o(\psi(C,\circ_i))}$ homogenous liftings of length $o(\psi(C,\circ_i))m$ in G^{ψ} of C for any integer $i, 1 \leq i \leq n$, where $\psi(C,\circ_i) = \psi(u_1,u_2) \circ_i \psi(u_2,u_3) \circ_i \cdots \circ_i \psi(u_{m-1},u_m) \circ_i \psi(u_m,u_1)$ and there are*

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\psi(C,\circ_i))}$$

homogenous liftings of C in G^{ψ} altogether.

Proof According to Theorem 4.2.2, there are liftings with initial vertex $(u_1)_a$ of C in G^{ψ} for $\forall a \in \widetilde{\Gamma}$. Whence, for any integer $i, 1 \leq i \leq n$, walks

$$\begin{aligned}
W_a &= (u_1)_a(u_2)_{a\circ_i\psi(u_1,u_2)} \cdots (u_m)_{a\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_{a\circ_i\psi(C,\circ_i)},\\
W_{a\circ_i\psi(C,\circ_i)} &= (u_1)_{a\circ_i\psi(C,\circ_i)}(u_2)_{a\circ_i\psi(C,\circ_i)\circ_i\psi(u_1,u_2)}\\
&\qquad \cdots (u_m)_{a\circ_i\psi(C,\circ_i)\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_{a\circ_i\psi^2(C,\circ_i)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
W_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)} &= (u_1)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)}(u_2)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)\circ_i\psi(u_1,u_2)}\\
&\qquad \cdots (u_m)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_a
\end{aligned}$$

are attached end-to-end to form a circuit of length $o(\psi(C,\circ_i))m$. Notice that there are $\dfrac{|\Gamma|}{o(\psi(C,\circ_i))}$ left cosets of the cyclic group generated by $\psi(C,\circ_i)$ in the group $(\Gamma,\circ_i)$ and each of them is correspondent with a homogenous lifting of C in G^{ψ}. Therefore, we get

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\psi(C,\circ_i))}$$

homogenous liftings of C in G^{ψ}. □

Corollary 4.2.3 *Let C be a k-circuit in a voltage graph (G,ψ) such that the order of $\psi(C,\circ)$ is m in the voltage group $(\Gamma;\circ)$. Then each component of the preimage $p^{-1}(C)$ is a km-circuit, and there are $\dfrac{|\Gamma|}{m}$ such components.*

The lifting G^{ζ} of a multi-voltage graph (G,ζ) of type I has a natural decomposition described in the following.

Theorem 4.2.4 *Let* $(G,\zeta), \zeta : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$, *be a multi-voltage graph of type I. Then*

$$G^{\zeta} = \bigoplus_{i=1}^{n} H_i,$$

where H_i *is an induced subgraph* $\langle E_i \rangle$ *of* G^{ζ} *for an integer* $i, 1 \le i \le n$ *with*

$$E_i = \{(u_a, v_{a\circ_i b}) | a, b \in \Gamma_i \text{ and } (u,v) \in E(G), \zeta(u,v) = b\}.$$

4.2.2 Subaction of Multi-Group. For a finite multi-group $\left(\widetilde{\Gamma}; O\right)$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$, $O = \{\circ_i, 1 \le i \le n\}$ and a graph G, if there exists a decomposition $G = \bigoplus\limits_{j=1}^{n} H_i$ and we can associate each element $g_i \in \Gamma_i$ a homeomorphism φ_{g_i} on the vertex set $V(H_i)$ for any integer $i, 1 \le i \le n$ such that

(1) $\varphi_{g_i \circ_i h_i} = \varphi_{g_i} \times \varphi_{h_i}$ for all $g_i, h_i \in \Gamma_i$, where "×" is an operation between homeomorphisms;

(2) φ_{g_i} is the identity homeomorphism if and only if g_i is the identity element of the group $(\Gamma_i; \circ_i)$,

then we say this association to be a *subaction of multi-group* $\widetilde{\Gamma}$ *on graph* G. If there exists a subaction of $\widetilde{\Gamma}$ on G such that $\varphi_{g_i}(u) = u$ only if $g_i = \mathbf{1}_{\Gamma_i}$ for any integer $i, 1 \le i \le n$, $g_i \in \Gamma_i$ and $u \in V_i$, we call it to be a *fixed-free subaction*.

A *left subaction lA* of $\widetilde{\Gamma}$ on G^{ψ} is defined by

For any integer $i, 1 \le i \le n$, *let* $V_i = \left\{u_a | u \in V(G), a \in \widetilde{\Gamma}\right\}$ *and* $g_i \in \Gamma_i$. *Define* $lA(g_i)(u_a) = u_{g_i \circ_i a}$ *if* $a \in V_i$. *Otherwise,* $g_i(u_a) = u_a$.

Then the following result holds.

Theorem 4.2.5 *Let* (G,ψ) *be a multi-voltage graph with* $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *and* $G = \bigoplus\limits_{j=1}^{n} H_i$ *with* $H_i = \langle E_i \rangle$, $1 \le i \le n$, *where* $E_i = \{(u_a, v_{a\circ_i b}) | a, b \in \Gamma_i \text{ and } (u,v) \in E(G), \zeta(u,v) = b\}$. *Then for any integer* $i, 1 \le i \le n$,

(1) *For* $\forall g_i \in \Gamma_i$, *the left subaction* $lA(g_i)$ *is a fixed-free subaction of an automorphism of* H_i;

(2) Γ_i *is an automorphism group of* H_i.

Proof Notice that $lA(g_i)$ is a one-to-one mapping on $V(H_i)$ for any integer $i, 1 \leq i \leq n$, $\forall g_i \in \Gamma_i$. By the definition of a lifting, an edge in H_i has the form $(u_a, v_{a\circ_i b})$ if $a, b \in \Gamma_i$. Whence,

$$(lA(g_i)(u_a), lA(g_i)(v_{a\circ_i b})) = (u_{g_i\circ_i a}, v_{g_i\circ_i a\circ_i b}) \in E(H_i).$$

As a result, $lA(g_i)$ is an automorphism of the graph H_i.

Notice that $lA : \Gamma_i \to \mathrm{Aut}H_i$ is an injection from Γ_i to $\mathrm{Aut}G^{\psi}$. Since $lA(g_i) \neq lA(h_i)$ for $\forall g_i, h_i \in \Gamma_i, g_i \neq h_i, 1 \leq i \leq n$. Otherwise, if $lA(g_i) = lA(h_i)$ for $\forall a \in \Gamma_i$, then $g_i \circ_i a = h_i \circ_i a$. Whence, $g_i = h_i$, a contradiction. Therefore, Γ_i is an automorphism group of H_i. Now for any integer i, $1 \leq i \leq n$, $g_i \in \Gamma_i$, it is implied by definition that $lA(g_i)$ is a fixed-free subaction on G^{ψ}. This completes the proof. □

Corollary 4.2.4 *Let* (G, α) *be a voltage graph with* $\alpha : X_{\frac{1}{2}}(G) \to \Gamma$. *Then* Γ *is an automorphism group of* G^{α}.

For a finite multi-group $\left(\widetilde{\Gamma}; O\right)$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ action on a graph $\widetilde{G}$, the vertex orbit $orb(v)$ of a vertex $v \in V(\widetilde{G})$ and the edge orbit $orb(e)$ of an edge $e \in E(\widetilde{G})$ are respectively defined by

$$orb(v) = \{g(v)|g \in \widetilde{\Gamma}\} \quad \text{and} \quad orb(e) = \{g(e)|g \in \widetilde{\Gamma}\}.$$

Then the *quotient graph* $\widetilde{G}/\widetilde{\Gamma}$ of $\widetilde{G}$ under the action of $\widetilde{\Gamma}$ is defined by

$$\begin{aligned}
&V(\widetilde{G}/\widetilde{\Gamma}) = \left\{ orb(v) \mid v \in V\left(\widetilde{G}\right) \right\},\\
&E(\widetilde{G}/\widetilde{\Gamma}) = \left\{ orb(e) \mid e \in E\left(\widetilde{G}\right) \right\},\\
&I(orb(e)) = (orb(u), orb(v)) \text{ if there exists } (u, v) \in E\left(\widetilde{G}\right).
\end{aligned}$$

For example, a quotient graph is shown in Fig.4.2.2, where, $\widetilde{\Gamma} = Z_5$.

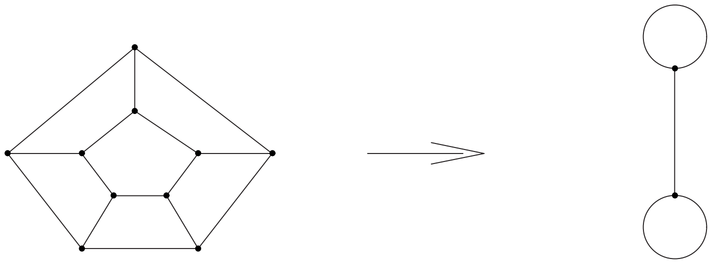


Fig 4.2.2

Then we get a necessary and sufficient condition for the lifting of a multi-voltage graph following.

Theorem 4.2.6 *If the subaction* $\mathscr{A}$ *of a finite multi-group* $\left(\widetilde{\Gamma}; O\right)$ *with* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *on a graph* $\widetilde{G} = \bigoplus\limits_{i=1}^{n} H_i$ *is fixed-free, then there is a multi-voltage graph* $\left(\widetilde{G}/\widetilde{\Gamma}, \zeta\right)$, $\zeta : X_{\frac{1}{2}}\left(\widetilde{G}/\widetilde{\Gamma}\right) \to \widetilde{\Gamma}$ *of type I such that*

$$\widetilde{G} \simeq \left(\widetilde{G}/\widetilde{\Gamma}\right)^{\zeta}.$$

Proof First, we choose positive directions for edges of $\widetilde{G}/\widetilde{\Gamma}$ and $\widetilde{G}$ so that the quotient map $q_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/\widetilde{\Gamma}$ is direction-preserving and that the action $\mathscr{A}$ of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions. Next, for any integer $i, 1 \leq i \leq n$ and $\forall v \in V\left(\widetilde{G}/\widetilde{\Gamma}\right)$, label one vertex of the orbit $q_{\widetilde{\Gamma}}^{-1}(v)$ in $\widetilde{G}$ as $v_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathscr{A}(g_i)(v_{1_{\Gamma_i}})$ as v_{g_i}. Now if the edge e of $\widetilde{G}/\widetilde{\Gamma}$ runs from u to w, we assigns the label e_{g_i} to the edge of the orbit $q_{\Gamma_i}^{-1}(e)$ that originates at the vertex u_{g_i}. Since Γ_i acts freely on H_i, there are just $|\Gamma_i|$ edges in the orbit $q_{\Gamma_i}^{-1}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $q_{\Gamma_i}^{-1}(v)$. Thus the choice of an edge to be labeled e_{g_i} is unique for any integer $i, 1 \leq i \leq n$. Finally, if the terminal vertex of the edge $e_{1_{\Gamma_i}}$ is w_{h_i}, one assigns a voltage h_i to the edge e in the quotient $\widetilde{G}/\widetilde{\Gamma}$, which enables us to get a multi-voltage graph $\left(\widetilde{G}/\widetilde{\Gamma}, \zeta\right)$. To show that this labeling of edges in $q_{\Gamma_i}^{-1}(e)$ and the choice of voltages $h_i, 1 \leq i \leq n$ for the edge e really yields an isomorphism $\vartheta : \widetilde{G} \to \left(\widetilde{G}/\widetilde{\Gamma}\right)^{\zeta}$, one needs to show that for $\forall g_i \in \Gamma_i, 1 \leq i \leq n$ that the edge e_{g_i} terminates at the vertex $w_{g_i \circ_i h_i}$. However, since $e_{g_i} = \mathscr{A}(g_i)(e_{1_{\Gamma_i}})$, the terminal vertex of the edge e_{g_i} must be the terminal vertex of the edge $\mathscr{A}(g_i)(e_{1_{\Gamma_i}})$, which is

$$\mathscr{A}(g_i)(w_{h_i}) = \mathscr{A}(g_i)\mathscr{A}(h_i)(w_{1_{\Gamma_i}}) = \mathscr{A}(g_i \circ_i h_i)(w_{1_{\Gamma_i}}) = w_{g_i \circ_i h_i}.$$

Under this labeling process, the isomorphism $\vartheta : \widetilde{G} \to \left(\widetilde{G}/\widetilde{\Gamma}\right)^{\zeta}$ identifies orbits in $\widetilde{G}$ with fibers of G^{ζ}. Moreover, it is defined precisely so that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ is consistent with the left subaction lA on the lifting graph G^{ζ}. □

Particularly, if $\left(\widetilde{\Gamma}; O\right)$ is a finite group, we get Theorem 4.1.5 as a corollary.

Corollary 4.2.5 *Let* $(\Gamma; \circ)$ *be a group acting freely on a graph* $\widetilde{G}$ *and let* G *be the resulting quotient graph. Then there is a voltage assignment* $\alpha : G \to \Gamma$ *and a labeling of the vertices* $\widetilde{G}$ *by the elements of* $V(G) \times \Gamma$ *such that* $\widetilde{G} = G^{\alpha}$ *and the given action of* $(\Gamma; \circ)$ *on* $\widetilde{G}$ *is the natural action of* $(\Gamma; \circ)$ *on* G^{α}.

§4.3 MULTI-VOLTAGE GRAPHS–TYPE II

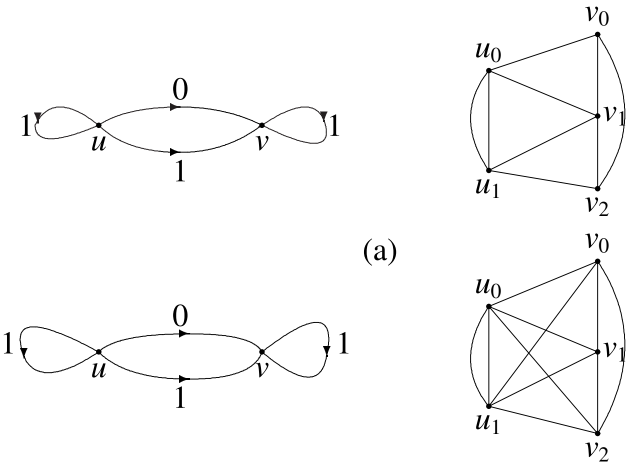
4.3.1 Multi-Voltage Graph of Type II. The multi-voltage graphs of type I are globally labeling edges by elements in finite multi-groups. Certainly, we can locally label edges in a graph by elements in groups. Thus the multi-voltage graphs of type II, formally defined in the following.

Definition 4.3.1 *Let* $\left(\widetilde{\Gamma}, O\right)$ *be a finite multi-group with* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$, $O = \{\circ_i;\ 1 \leq i \leq n\}$ *and let* G *be a graph with vertices partition* $V(G) = \bigcup\limits_{i=1}^{n} V_i$. *For any integers* $i, j, 1 \leq i, j \leq n$, *if there is a mapping* $\tau : X_{\frac{1}{2}}\left(\left\langle E_G(V_i, V_j)\right\rangle\right) \rightarrow \Gamma_i \bigcap \Gamma_j$ *and* $\varsigma : V_i \rightarrow \Gamma_i$ *such that* $\tau(e^{-1}) = (\tau(e^+))^{-1}$ *for* $\forall e^+ \in X_{\frac{1}{2}}(G)$ *and the vertex subset* V_i *is associated with the group* $(\Gamma_i, \circ_i)$ *for any integer* $i, 1 \leq i \leq n$, *then* (G, τ, ς) *is called a multi-voltage graph of type II.*

The lifting of a multi-voltage graph (G, τ, ς) of type II is defined in the following.

Definition 4.3.2 *For a multi-voltage graph* (G, τ, ς) *of type II, the lifting graph* $G^{(\tau,\varsigma)} = \left(V\left(G^{(\tau,\varsigma)}\right),\ E\left(G^{(\tau,\varsigma)}\right); I\left(G^{(\tau,\varsigma)}\right)\right)$ *of* (G, τ, ς) *is defined by*

$$\begin{aligned}
&V\left(G^{(\tau,\varsigma)}\right) = \bigcup_{i=1}^{n} \{V_i \times \Gamma_i\},\\
&E\left(G^{(\tau,\varsigma)}\right) = \left\{(u_a, v_{a\circ b}) | e^+ = (u, v) \in X_{\frac{1}{2}}(G), \psi(e^+) = b, a \circ b \in \widetilde{\Gamma}\right\},\\
&I\left(G^{(\tau,\varsigma)}\right) = \left\{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^{(\tau,\varsigma)})\right\}.
\end{aligned}$$



(a)

(b)

Fig 4.3.1

Two multi-voltage graphs of type II with their lifting are shown in (a) and (b) of Fig.4.3.1, where $\widetilde{\Gamma} = Z_2 \bigcup Z_3$, $V_1 = \{u\}$, $V_2 = \{v\}$ and $\varsigma : V_1 \to Z_2$, $\varsigma : V_2 \to Z_3$.

Theorem 4.3.1 *Let (G, τ, ς) be a multi-voltage graph of type II and let $W_k = u_1u_2 \cdots u_k$ be a walk in G. Then there exists a lifting of $W^{(\tau,\varsigma)}$ with an initial vertex $(u_1)_a, a \in \varsigma^{-1}(u_1)$ in $G^{(\tau,\varsigma)}$ if and only if $a \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$ and for any integer $s, 1 \le s < k$, $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \tau(u_2u_3) \circ_{i_3} \cdots \circ_{i_{s-1}} \tau(u_{s-2}u_{s-1}) \in \varsigma^{-1}(u_{s-1}) \bigcap \varsigma^{-1}(u_s)$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_{j+1})$ for any integer $j, 1 \le j \le s$.*

Proof By the definition of the lifting of a multi-voltage graph of type II, there exists a lifting of the edge u_1u_2 in $G^{(\tau,\varsigma)}$ if and only if $a \circ_{i_1} \tau(u_1u_2) \in \varsigma^{-1}(u_2)$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_2)$. Since $\tau(u_1u_2) \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$, we get that $a \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$. Similarly, there exists a lifting of the subwalk $W_2 = u_1u_2u_3$ in $G^{(\tau,\varsigma)}$ if and only if $a \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$ and $a \circ_{i_1} \tau(u_1u_2) \in \varsigma^{-1}(u_2) \bigcap \varsigma^{-1}(u_3)$.

Now assume there exists a lifting of the subwalk $W_l = u_1u_2u_3 \cdots u_l$ in $G^{(\tau,\varsigma)}$ if and only if $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{t-2}} \tau(u_{t-2}u_{t-1}) \in \varsigma^{-1}(u_{t-1}) \bigcap \varsigma^{-1}(u_t)$ for any integer $t, 1 \le t \le l$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_{j+1})$ for any integer $j, 1 \le j \le l$. We consider the lifting of the subwalk $W_{l+1} = u_1u_2u_3 \cdots u_{l+1}$. Notice that if there exists a lifting of the subwalk W_l in $G^{(\tau,\varsigma)}$, then the terminal vertex of W_l in $G^{(\tau,\varsigma)}$ is $(u_l)_{a\circ_{i_1}\tau(u_1u_2)\circ_{i_2}\cdots\circ_{i_{l-1}}\tau(u_{l-1}u_l)}$. We only need to find a necessary and sufficient condition for existing a lifting of u_lu_{l+1} with an initial vertex $(u_l)_{a\circ_{i_1}\tau(u_1u_2)\circ_{i_2}\cdots\circ_{i_{l-1}}}\tau(u_{l-1}u_l)$. By definition, there exists such a lifting of the edge u_lu_{l+1} if and only if $(a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}})\tau(u_{l-1}u_l)) \circ_l \tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$. Since $\tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$ by the definition of multi-voltage graphs of type II, we know that $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}} \tau(u_{l-1}u_l) \in \varsigma^{-1}(u_{l+1})$.

Continuing this process, we get the assertion by the induction principle. □

Corollary 4.3.1 *Let G a graph with vertices partition $V(G) = \bigcup_{i=1}^{n} V_i$ and let $(\Gamma; \circ)$ be a finite group, $\Gamma_i \prec \Gamma$ for any integer $i, 1 \le i \le n$. If (G, τ, ς) is a multi-voltage graph with $\tau : X_{\frac{1}{2}}(G) \to \Gamma$ and $\varsigma : V_i \to \Gamma_i$ for any integer $i, 1 \le i \le n$, then for a walk W in G with an initial vertex u, there exists a lifting $W^{(\tau,\varsigma)}$ in $G^{(\tau,\varsigma)}$ with the initial vertex $u_a, a \in \varsigma^{-1}(u)$ if and only if $a \in \bigcap_{v \in V(W)} \varsigma^{-1}(v)$.*

Similarly, if $\Gamma_i = \Gamma$ and $V_i = V(G)$ for any integer $i, 1 \le i \le n$, the number of liftings of a walk in a multi-voltage graph of type II can be determined.

Theorem 4.3.2 *Let $\left(\widetilde{\Gamma}; O\right)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup_{i=1}^{n} \Gamma$, $O = \{\circ_i; 1 \le i \le n\}$ and* *let* $W = e^1e^2\cdots e^k$ *be a walk with an initial vertex* u *in a multi-voltage graph* (G,τ,ς), $\tau : X_{\frac{1}{2}}(G) \rightarrow \bigcap_{i=1}^{n}\Gamma$ *and* $\varsigma : V(G) \rightarrow \Gamma$, *of type II. Then there are* n^k *liftings of* W *in* $G^{(\tau,\varsigma)}$ *with an initial vertex* u_a *for* $\forall a \in \widetilde{\Gamma}$.

Proof The proof is similar to that of Theorem 4.2.3. □

Theorem 4.3.3 *Let* $(\widetilde{\Gamma}; O)$ *be a finite multi-group with* $\widetilde{\Gamma} = \bigcup_{i=1}^{n}\Gamma$, $O = \{\circ_i; 1 \leq i \leq n\}$ *and let* $C = u_1u_2\cdots u_mu_1$ *be a circuit in a multi-voltage graph* (G,τ,ς), *where* $\tau : X_{\frac{1}{2}}(G) \rightarrow \bigcap_{i=1}^{n}\Gamma$ *and* $\varsigma : V(G) \rightarrow \Gamma$. *Then there are* $\dfrac{|\Gamma|}{o(\tau(C,\circ_i))}$ *liftings of length* $o(tau(C,\circ_i))m$ *in* $G^{(\tau,\varsigma)}$ *of* C *for any integer* $i, 1 \leq i \leq n$, *where* $\tau(C,\circ_i) = \tau(u_1,u_2)\circ_i \tau(u_2,u_3)\circ_i\cdots\circ_i\tau(u_{m-1},u_m)\circ_i \tau(u_m,u_1)$, *and there are*

$$\sum_{i=1}^{n}\frac{|\Gamma|}{o(\tau(C,\circ_i))}$$

liftings of C *in* $G^{(\tau,\varsigma)}$ *altogether.*

Proof The proof is similar to that of Theorem 4.2.3. □

4.3.2 Subgraph Isomorphism. Let G_1, G_2 be graph and H a subgraph of G_1 and G_2. We introduce the conception of H-isomorphism of graph following.

Definition 4.3.3 *Let* G_1, G_2 *be two graphs and* H *a subgraph of* G_1 *and* G_2. *A one-to-one mapping* ξ *between* G_1 *and* G_2 *is called an* H*-isomorphism if for any subgraph* J *isomorphic to* H *in* G_1, $\xi(J)$ *is also a subgraph isomorphic to* H *in* G_2.

If $G_1 = G_2 = G$, *then an* H*-isomorphism between* G_1 *and* G_2 *is called an* H*-automorphism of* G. *Certainly, all* H*-automorphisms form a group under the composition operation, denoted by* $\mathrm{Aut}_H G$ *and* $\mathrm{Aut}_H G = \mathrm{Aut} G$ *if we take* $H = K_2$.

For example, let $H = \langle E(x, N_G(x))\rangle$ for $\forall x \in V(G)$. Then the H-automorphism group of a complete bipartite graph $K(n,m)$ is $\mathrm{Aut}_H K(n,m) = S_n[S_m] = \mathrm{Aut}K(n,m)$. There H-automorphisms are called *star-automorphisms*.

Theorem 4.3.4 *Let* G *be a graph. If there is a decomposition* $G = \bigoplus_{i=1}^{n} H_i$ *with* $H_i \simeq H$ *for* $1 \leq i \leq n$ *and* $H = \bigoplus_{j=1}^{m} J_j$ *with* $J_j \simeq J$ *for* $1 \leq j \leq m$, *then*

(1) $\langle \iota_i, \iota_i : H_1 \rightarrow H_i$, *an isomorphism*, $1 \leq i \leq n\rangle = S_n \preceq \mathrm{Aut}_H G$, *and particularly,* $S_n \preceq \mathrm{Aut}_H K_{2n+1}$ *if* $H = C$, *a hamiltonian circuit in* K_{2n+1}.

(2) $\mathrm{Aut}_J G \preceq \mathrm{Aut}_H G$*, and particularly,* $\mathrm{Aut}G \preceq \mathrm{Aut}_H G$ *for a simple graph* G*.*

Proof (1) For any integer $i, 1 \leq i \leq n$, we prove there is a such H-automorphism ι on G that $\iota_i : H_1 \rightarrow H_i$. In fact, since $H_i \simeq H$, $1 \leq i \leq n$, there is an isomorphism $\theta : H_1 \rightarrow H_i$. We define ι_i as follows:

$$\iota_i(e) = \begin{cases} \theta(e), & \text{if } e \in V(H_1) \bigcup E(H_1), \\ e, & \text{if} e \in (V(G) \setminus V(H_1)) \bigcup (E(G) \setminus E(H_1)). \end{cases}$$

Then ι_i is a one-to-one mapping on the graph G and is also an H-isomorphism by definition. Whence,

$$\langle \iota_i, \iota_i : H_1 \rightarrow H_i, \text{ an isomorphism, } 1 \leq i \leq n \rangle \preceq \mathrm{Aut}_H G.$$

Since $\langle \iota_i, 1 \leq i \leq n \rangle \simeq \langle (1, i), 1 \leq i \leq n \rangle = S_n$, thereby we get that $S_n \preceq \mathrm{Aut}_H G$.

For a complete graph K_{2n+1}, we know its a decomposition $K_{2n+1} = \bigoplus\limits_{i=1}^{n} C_i$ with

$$C_i = v_0 v_i v_{i+1} v_{i-1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0$$

for any integer $i, 1 \leq i \leq n$ by Theorem 2.4.2. Whence, we get that

$$S_n \preceq \mathrm{Aut}_H K_{2n+1}$$

if we choose a hamiltonian circuit H in K_{2n+1}.

(2) Choose $\sigma \in \mathrm{Aut}_J G$. By definition, for any subgraph A of G, if $A \simeq J$, then $\sigma(A) \simeq J$. Notice that $H = \bigoplus\limits_{j=1}^{m} J_j$ with $J_j \simeq J$ for $1 \leq j \leq m$. Therefore, for any subgraph $B, B \simeq H$ of G, $\sigma(B) \simeq \bigoplus\limits_{j=1}^{m} \sigma(J_j) \simeq H$. This fact implies that $\sigma \in \mathrm{Aut}_H G$.

Notice that for a simple graph G, we have a decomposition $G = \bigoplus\limits_{i=1}^{\varepsilon(G)} K_2$ and $\mathrm{Aut}_{K_2} G = \mathrm{Aut}G$. Whence, $\mathrm{Aut}G \preceq \mathrm{Aut}_H G$. □

The equality in Theorem 4.3.4(2) does not always hold. For example, a one-to-one mapping σ on the lifting graph of Fig.4.3.2(a): $\sigma(u_0) = u_1$, $\sigma(u_1) = u_0$, $\sigma(v_0) = v_1$, $\sigma(v_1) = v_2$ and $\sigma(v_2) = v_0$ is not an automorphism, but it is an H-automorphism with H being a star $S_{1.2}$.

For automorphisms of the lifting $G^{(\tau,\varsigma)}$ of a multi-voltage graph (G, τ, ς) of type II, we get a result following.

Theorem 4.3.5 *Let (G, τ, ς) be a multi-voltage graph of type II with $\tau : X_{\frac{1}{2}}(G) \rightarrow \bigcap\limits_{i=1}^{n} \Gamma_i$ and $\varsigma : V_i \rightarrow \Gamma_i$. Then for any integers $i, j, 1 \leq i, j \leq n$,*

(1) *for $\forall g_i \in \Gamma_i$, the left action $lA(g_i)$ on $\langle V_i \rangle^{(\tau,\varsigma)}$ is a fixed-free action of an automorphism of $\langle V_i \rangle^{(\tau,\varsigma)}$;*

(2) *for $\forall g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ on $\left\langle E_G(V_i, V_j) \right\rangle^{(\tau,\varsigma)}$ is a star-automorphism of $\left\langle E_G(V_i, V_j) \right\rangle^{(\tau,\varsigma)}$.*

Proof The proof of (1) is similar to that of Theorem 4.2.4. We prove the assertion (2). A star with a central vertex u_a, $u \in V_i, a \in \Gamma_i \bigcap \Gamma_j$ is the graph $S_{star} = \left\langle \{(u_a, v_{a \circ_j b}) \text{ if } (u, v) \in E_G(V_i, V_j), \tau(u, v) = b\} \right\rangle$. By definition, the left action $lA(g_{ij})$ is a one-to-one mapping on $\left\langle E_G(V_i, V_j) \right\rangle^{(\tau,\varsigma)}$. Now for any element $g_{ij}, g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ of g_{ij} on a star S_{star} is

$$lA(g_{ij})(S_{star}) = \left\langle \left\{ (u_{g_{ij} \circ_i a}, v_{(g_{ij} \circ_i a) \circ_j b}) \text{ if } (u, v) \in E_G(V_i, V_j), \tau(u, v) = b \right\} \right\rangle = S_{star}.$$

Whence, $lA(g_{ij})$ is a star-automorphism of $\left\langle E_G(V_i, V_j) \right\rangle^{(\tau,\varsigma)}$. □

Let $\widetilde{G}$ be a graph and let $\left(\widetilde{\Gamma}; O\right)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $O = \{\circ_i;\ 1 \leq i \leq n\}$. If there is a partition for the vertex set $V\left(\widetilde{G}\right) = \bigcup\limits_{i=1}^{n} V_i$ such that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ consists of Γ_i action on $\langle V_i \rangle$ and $\Gamma_i \bigcap \Gamma_j$ on $\left\langle E_G(V_i, v_j) \right\rangle$ for $1 \leq i, j \leq n$, we call such an action to be a *partially-action*. A partially-action is called *fixed-free* if Γ_i is fixed-free on $\langle V_i \rangle$ and the action of each element in $\Gamma_i \bigcap \Gamma_j$ is a star-automorphism and fixed-free on $\left\langle E_G(V_i, V_j) \right\rangle$ for any integers $i, j, 1 \leq i, j \leq n$. These orbits of a partially-action are defined to be

$$orb_i(v) = \{g(v) | g \in \Gamma_i, v \in V_i\}$$

for any integer $i, 1 \leq i \leq n$ and

$$orb(e) = \left\{ g(e) | e \in E(\widetilde{G}), g \in \widetilde{\Gamma} \right\}.$$

A *partially-quotient graph* $\widetilde{G}/_p\widetilde{\Gamma}$ is defined by

$$V\left(\widetilde{G}/_p\widetilde{\Gamma}\right) = \bigcup_{i=1}^{n} \{\, orb_i(v) \mid v \in V_i \}, \quad E\left(\widetilde{G}/_p\widetilde{\Gamma}\right) = \left\{ orb(e) | e \in E(\widetilde{G}) \right\}$$

and $I\left(\widetilde{G}/_p\widetilde{\Gamma}\right) = \{I(e) = (orb_i(u), orb_j(v)) \text{ if } u \in V_i, v \in V_j \text{ and } (u, v) \in E\left(\widetilde{G}\right), 1 \leq i, j \leq n\}$. For example, a partially-quotient graph is shown in Fig.4.3.2, where $V_1 = \{u_0, u_1, u_2, u_3\}$, $V_2 = \{v_0, v_1, v_2\}$ and $\Gamma_1 = Z_4$, $\Gamma_2 = Z_3$.

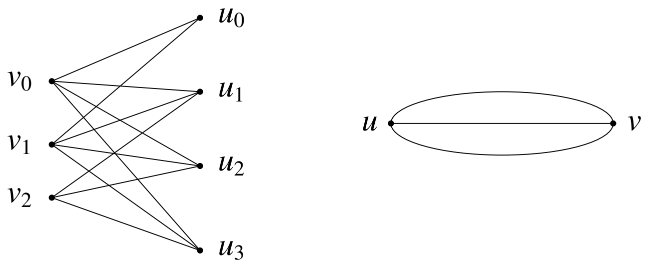


Fig 4.3.2

We get a necessary and sufficient condition for the lifting of a multi-voltage graph of type II following.

Theorem 4.3.6 *If the partially-action $\mathcal{P}_a$ of a finite multi-group $\left(\widetilde{\Gamma}; O\right)$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n}\Gamma_i$ and $O = \{\circ_i; 1 \leq i \leq n\}$ on a graph $\widetilde{G}$ with $V\left(\widetilde{G}\right) = \bigcup\limits_{i=1}^{n} V_i$ is fixed-free, then there is a multi-voltage graph $\left(\widetilde{G}/_p\Gamma, \tau, \varsigma\right)$, $\tau : X_{\frac{1}{2}}(\widetilde{G}/\widetilde{\Gamma}) \to \widetilde{\Gamma}$, $\varsigma : V_i \to \Gamma_i$ of type II such that*

$$\widetilde{G} \simeq \left(\widetilde{G}/_p\widetilde{\Gamma}\right)^{(\tau,\varsigma)}.$$

Proof Similar to the proof of Theorem 4.2.6, we also choose positive directions on these edges of $\widetilde{G}/_p\widetilde{\Gamma}$ and $\widetilde{G}$ so that the partially-quotient map $p_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/_p\widetilde{\Gamma}$ is direction-preserving and the partially-action of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions.

For any integer $i, 1 \leq i \leq n$ and $\forall v^i \in V_i$, we can label v^i as $v^i_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathcal{P}_a(g_i)((v_i)_{1_{\Gamma_i}})$ as $v^i_{g_i}$. Now if the edge e of $\widetilde{G}/_p\widetilde{\Gamma}$ runs from u to w, we assign the label e_{g_i} to the edge of the orbit $p^{-1}(e)$ that originates at the vertex $u^i_{g_i}$ and terminates at $w^j_{h_j}$.

Since Γ_i acts freely on $\langle V_i\rangle$, there are just $|\Gamma_i|$ edges in the orbit $p^{-1}_{\Gamma_i}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $p^{-1}_{\Gamma_i}(v)$. Thus for any integer $i, 1 \leq i \leq n$, the choice of an edge in $p^{-1}(e)$ to be labeled e_{g_i} is unique. Finally, if the terminal vertex of the edge e_{g_i} is $w^j_{h_j}$, one assigns voltage $g_i^{-1} \circ_j h_j$ to the edge e in the partially-quotient graph $\widetilde{G}/_p\widetilde{\Gamma}$ if $g_i, h_j \in \Gamma_i \bigcap \Gamma_j$ for $1 \leq i, j \leq n$.

Under this labeling process, the isomorphism $\vartheta : \widetilde{G} \to \left(\widetilde{G}/_p\widetilde{\Gamma}\right)^{(\tau,\varsigma)}$ identifies orbits in $\widetilde{G}$ with fibers of $G^{(\tau,\varsigma)}$. □

The multi-voltage graphs defined in Sections 4.2 and 4.3 enables us to enlarge the application field of voltage graphs. For example, a complete bipartite graph $K(n, m)$ is a lifting of a multi-voltage graph, but it is not a lifting of a voltage graph in general if $n \neq m$.

§4.4 MULTI-SPACES ON GRAPHS

4.4.1 Graph Model. A graph is called a *directed graph* if there is an orientation on its every edge. A directed graph $\overrightarrow{G}$ is called an *Euler graph* if we can travel all edges of $\overrightarrow{G}$ alone orientations on its edges with no repeat starting at any vertex $u \in V(\overrightarrow{G})$ and come back to u. For a directed graph $\overrightarrow{G}$, we use the convention that the orientation on the edge e is $u \to v$ for $\forall e = (u, v) \in E(\overrightarrow{G})$ and say that e is *incident from* u and *incident to* v. For $u \in V(\overrightarrow{G})$, the *outdegree* $\rho^{+}_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident from u and the *indegree* $\rho^{-}_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident to u. Whence, we know that

$$\rho^{+}_{\overrightarrow{G}}(u) + \rho^{-}_{\overrightarrow{G}}(u) = \rho_{\overrightarrow{G}}(u).$$

It is well-known that a graph $\overrightarrow{G}$ is Eulerian if and only if $\rho^{+}_{\overrightarrow{G}}(u) = \rho^{-}_{\overrightarrow{G}}(u)$ for $\forall u \in V\left(\overrightarrow{G}\right)$, seeing examples in [11] for details. For a multiple 2-edge (a, b), if two orientations on edges are both to a or both to b, then we say it to be a *parallel multiple* 2*-edge*. If one orientation is to a and another is to b, then we say it to be an *opposite multiple* 2*-edge*.

Now let $(A; \circ)$ be an algebraic system with operation "$\circ$". We associate a *weighted graph* $G[A]$ for $(A; \circ)$ defined as in the next definition.

Definition 4.4.1 *Let* $(A; \circ)$ *be an algebraic system. Define a weighted graph* $G[A]$ *associated with* $(A; \circ)$ *by*

$$V(G[A]) \ = \ A$$

and

$$E(G[A]) = \{(a, c) \ with \ weight \ \circ b \mid if \ a \circ b = c \ for \ \forall a, b, c \in A\}$$

as shown in Fig.4.4.1.

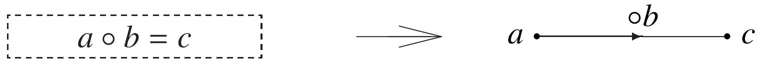


Fig.4.4.1

For example, the associated graph $G[Z_4]$ for commutative group Z_4 is shown in Fig.4.4.2.

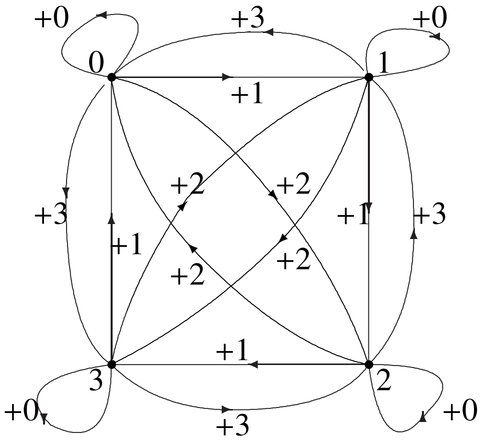


Fig.4.4.2

4.4.2 Graph Model Property. The advantage of Definition 4.4.1 is that for any edge with end-vertices a, c in $G[A]$, if its weight is $\circ b$, then a$\circ b = c$ and vice versa. Furthermore, if $a \circ b = c$, then there is one and only one edge in $G[A]$ with vertices a, c and weight $\circ b$. This property enables us to find some structure properties of $G[A]$ for an algebraic system $(A; \circ)$.

P1. *$G[A]$ is connected if and only if there are no partition $A = A_1 \bigcup A_2$ such that for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A; \circ)$.*

If $G[A]$ is disconnected, we choose one component C and let $A_1 = V(C)$. Define $A_2 = V(G[A]) \setminus V(C)$. Then we get a partition $A = A_1 \bigcup A_2$ and for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A; \circ)$, a contradiction and vice versa.

P2. *If there is a unit $\mathbf{1}_A$ in $(A; \circ)$, then there exists a vertex $\mathbf{1}_A$ in $G[A]$ such that the weight on the edge $(\mathbf{1}_A, x)$ is $\circ x$ if $\mathbf{1}_A \circ x$ is defined in $(A; \circ)$ and vice versa.*

P3. *For $\forall a \in A$, if a^{-1} exists, then there is an opposite multiple 2-edge $(\mathbf{1}_A, a)$ in $G[A]$ with weights $\circ a$ and $\circ a^{-1}$, respectively and vice versa.*

P4. *For $\forall a, b \in A$ if $a \circ b = b \circ a$, then there are edges (a, x) and (b, x), $x \in A$ in $(A; \circ)$ with weights $w(a, x) = \circ b$ and $w(b, x) = \circ a$, respectively and vice versa.*

P5. *If the cancellation law holds in $(A; \circ)$, i.e., for $\forall a, b, c \in A$, if $a \circ b = a \circ c$ then $b = c$, then there are no parallel multiple 2-edges in $G[A]$ and vice versa.*

The properties P2,P3,P4 and P5 are gotten by definition immediately. Each of these cases is shown in Fig.4.4.3(1), (2), (3) and (4), respectively.

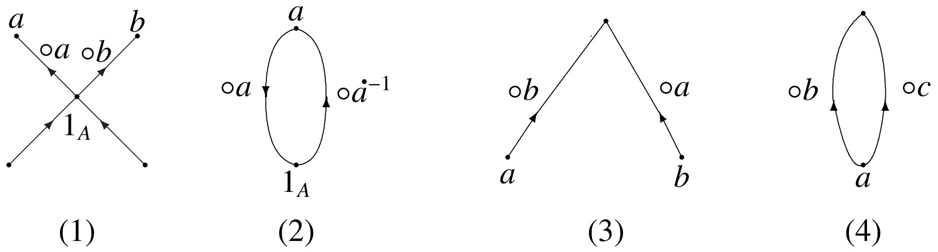


Fig.4.4.3

Definition 4.4.2 *An algebraic system* $(A; \circ)$ *is called to be a one-way system if there exists a mapping* $\varpi : A \to A$ *such that if* $a \circ b \in A$*, then there exists a unique* $c \in A$*,* $c \circ \varpi(b) \in A$*.* ϖ *is called a one-way function on* $(A; \circ)$*.*

We have the following results for an algebraic system $(A; \circ)$ with its associated weighted graph $G[A]$.

Theorem 4.4.1 *Let* $(A; \circ)$ *be an algebraic system with a associated weighted graph* $G[A]$*. Then*

(1) *If there is a one-way function* ϖ *on* $(A; \circ)$*, then* $G[A]$ *is an Euler graph, and vice versa, if* $G[A]$ *is an Euler graph, then there exist a one-way function* ϖ *on* $(A; \circ)$*.*

(2) *If* $(A; \circ)$ *is a complete algebraic system, then the outdegree of every vertex in* $G[A]$ *is* $|A|$*; in addition, if the cancellation law holds in* $(A; \circ)$*, then* $G[A]$ *is a complete multiple* 2*-graph with a loop attaching at each of its vertices such that each edge between two vertices in* $G[A]$ *is an opposite multiple* 2*-edge, and vice versa.*

Proof Let $(A; \circ)$ be an algebraic system with a associated weighted graph $G[A]$.

(1) Assume ϖ is a one-way function ϖ on $(A; \circ)$. By definition there exists $c \in A$, $c \circ \varpi(b) \in A$ for $\forall a \in A$, $a \circ b \in A$. Thereby there is a one-to-one correspondence between edges from a with edges to a. That is, $\rho^+_{G[A]}(a) = \rho^-_{G[A]}(a)$ for $\forall a \in V(G[A])$. Therefore, $G[A]$ is an Euler graph.

Now if $G[A]$ is an Euler graph, then there is a one-to-one correspondence between edges in $E^- = \{e_i^-; 1 \le i \le k\}$ from a vertex a with edges $E^+ = \{e_i^+; 1 \le i \le k\}$ to the vertex a. For any integer i, $1 \le i \le k$, define $\varpi : w(e_i^-) \to w(e_i^+)$. Therefore, ϖ is a well-defined one-way function on $(A; \circ)$.

(2) If $(A;\circ)$ is complete, then for $\forall a \in A$ and $\forall b \in A$, $a \circ b \in A$. Therefore, $\rho^{+}_{\overrightarrow{G}}(a) = |A|$ for any vertex $a \in V(G[A])$.

If the cancellation law holds in $(A;\circ)$, by $P5$ there are no parallel multiple 2-edges in $G[A]$. Whence, each edge between two vertices is an opposite 2-edge and weights on loops are $\circ\mathbf{1}_A$.

By definition, if $G[A]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge, we know that $(A;\circ)$ is a complete algebraic system with the cancellation law holding by the definition of $G[A]$. □

Corollary 4.4.1 *Let* Γ *be a semigroup. Then* $G[\Gamma]$ *is a complete multiple* 2*-graph with a loop attaching at each of its vertices such that each edge between two vertices in* $G[A]$ *is an opposite multiple* 2*-edge.*

Notice that in a group Γ, $\forall g \in \Gamma$, if $g^2 \neq \mathbf{1}_\Gamma$, then $g^{-1} \neq g$. Whence, all elements of order$>$ 2 in Γ can be classified into pairs. This fact enables us to know the following result.

Corollary 4.4.2 *Let* Γ *be a group of even order. Then there are opposite multiple* 2*-edges in* $G[\Gamma]$ *such that weights on its* 2 *directed edges are the same.*

4.4.3 Multi-Space on Graph. Let $\left(\widetilde{\Gamma};O\right)$ be an algebraic multi-space. Its associated weighted graph is defined in the following.

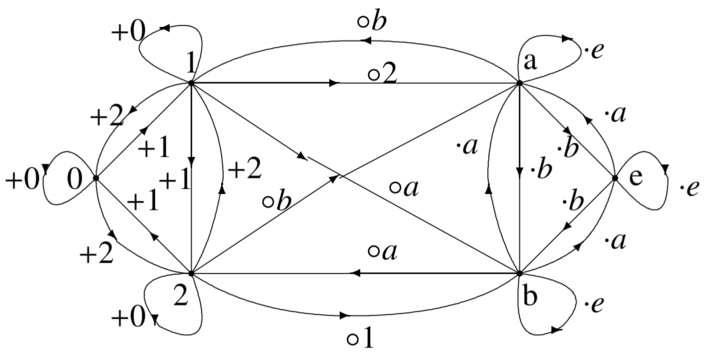


Fig.4.4.4

Definition 4.4.3 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be an algebraic multi-space with* $(\Gamma_i;\circ_i)$ *being an algebraic* *system for any integer i*, $1 \leq i \leq n$. *Define a weighted graph* $G\left(\widetilde{\Gamma}\right)$ *associated with* $\widetilde{\Gamma}$ *by*

$$G\left(\widetilde{\Gamma}\right) = \bigcup_{i=1}^{n} G[\Gamma_i],$$

where $G[\Gamma_i]$ *is the associated weighted graph of* $(\Gamma_i; \circ_i)$ *for* $1 \leq i \leq n$.

For example, the weighted graph shown in Fig.4.4.4 is correspondent with a multi-space $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $(\Gamma_1; +) = (Z_3, +)$, $\Gamma_2 = \{e, a, b\}$, $\Gamma_3 = \{1, 2, a, b\}$ and these operations "$\cdot$" on Γ_2 and "$\circ$" on Γ_3 are shown in tables 4.4.1 and 4.4.2.

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

table 4.4.1

$\circ$	1	2	a	b
1	*	a	b	*
2	b	*	*	a
a	*	*	*	1
b	*	*	2	*

table 4.4.2

Notice that the correspondence between the multi-space $\widetilde{\Gamma}$ and the weighted graph $G\left[\widetilde{\Gamma}\right]$ is one-to-one. We immediately get the following result.

Theorem 4.4.2 *The mappings* $\pi : \widetilde{\Gamma} \to G\left[\widetilde{\Gamma}\right]$ *and* $\pi^{-1} : G\left[\widetilde{\Gamma}\right] \to \widetilde{\Gamma}$ *are all one-to-one.*

According to Theorems 4.4.1 and 4.4.2, we get some consequences in the following.

Corollary 4.4.3 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a multi-space with an algebraic system* $(\Gamma_i; \circ_i)$ *for any integer i*, $1 \leq i \leq n$. *If for any integer i*, $1 \leq i \leq n$, $G[\Gamma_i]$ *is a complete multiple* 2*-graph with a loop attaching at each of its vertices such that each edge between two vertices in* $G[\Gamma_i]$ *is an opposite multiple* 2*-edge, then* $\widetilde{\Gamma}$ *is a complete multi-space.*

Corollary 4.4.4 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a multi-group with an operation set* $O\left(\widetilde{\Gamma}\right) = \{\circ_i; 1 \leq i \leq n\}$. *Then there is a partition* $G\left[\widetilde{\Gamma}\right] = \bigcup\limits_{i=1}^{n} G_i$ *such that each* G_i *being a complete multiple*

2-graph attaching with a loop at each of its vertices such that each edge between two vertices in $V(G_i)$ is an opposite multiple 2-edge for any integer $i, 1 \leq i \leq n$.

Corollary 4.4.5 *Let F be a body. Then $G[F]$ is a union of two graphs $K^2(F)$ and $K^2(F^*)$, where $K^2(F)$ or $K^2(F^*)$ is a complete multiple 2-graph with vertex set F or $F^* = F \setminus \{0\}$ and with a loop attaching at each of its vertices such that each edge between two different vertices is an opposite multiple 2-edge.*

4.4.4 Cayley Graph of Multi-Group. Similar to that of Cayley graphs of a finite generated group, we can also define *Cayley graphs of a finite generated multi-group*, where a multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ is said to be *finite generated* if the group Γ_i is finite generated for any integer $i, 1 \leq i \leq n$, i.e., $\Gamma_i = \langle x_i, y_i, \cdots, z_{s_i}\rangle$. We denote by $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n\rangle$ if $\widetilde{\Gamma}$ is finite generated by $\{x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n\}$.

Definition 4.4.4 *Let $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n\rangle$ be a finite generated multi-group, $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, where $1_{\Gamma_i} \notin S_i$, $\widetilde{S}^{-1} = \left\{a^{-1} | a \in \widetilde{S}\right\} = \widetilde{S}$ and $\langle S_i\rangle = \Gamma_i$ for any integer $i, 1 \leq i \leq n$. A Cayley graph $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ is defined by*

$$V\left(Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)\right) = \widetilde{\Gamma}$$

and

$$E\left(Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)\right) = \{(g, h) | \ if \ there \ exists \ an \ integer \ i, g^{-1} \circ_i h \in S_i, 1 \leq i \leq n\}.$$

By Definition 4.4.4, we immediately get the following result for Cayley graphs of a finite generated multi-group.

Theorem 4.4.3 *For a Cayley graph $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$,*

$$Cay\left(\widetilde{\Gamma} : \widetilde{S}\right) = \bigcup_{i=1}^{n} Cay(\Gamma_i : S_i).$$

It is well-known that *every Cayley graph of order≥ 3 is 2-connected*. But in general, a Cayley graph of a multi-group is not connected. For the connectedness of Cayley graphs of multi-groups, we get the following result.

Theorem 4.4.4 *A Cayley graph $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$ is connected if and only if for any integer $i, 1 \leq i \leq n$, there exists an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$.*

Proof According to Theorem 4.4.3, if there is an integer $i, 1 \leq i \leq n$ such that $\Gamma_i \bigcap \Gamma_j = \emptyset$ for any integer $j, 1 \leq j \leq n,\ j \neq i$, then there are no edges with the form $(g_i, h),\ g_i \in \Gamma_i,\ h \in \widetilde{\Gamma} \setminus \Gamma_i$. Thus $Cay\left(\Gamma : \widetilde{S}\right)$ is not connected.

Notice that $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right) = \bigcup\limits_{i=1}^{n} Cay(\Gamma_i : S_i)$. Not loss of generality, we assume that $g \in \Gamma_k$ and $h \in \Gamma_l$, where $1 \leq k, l \leq n$ for any two elements $g, h \in \widetilde{\Gamma}$. If $k = l$, then there must exists a path connecting g and h in $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$.

Now if $k \neq l$ and for any integer $i, 1 \leq i \leq n$, there is an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$, then we can find integers $i_1, i_2, \cdots, i_s,\ 1 \leq i_1, i_2, \cdots, i_s \leq n$ such that

$$\Gamma_k \bigcap \Gamma_{i_1} \neq \emptyset,$$

$$\Gamma_{i_1} \bigcap \Gamma_{i_2} \neq \emptyset,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\Gamma_{i_s} \bigcap \Gamma_l \neq \emptyset.$$

Therefore, we can find a path connecting g and h in $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ passing through these vertices in $Cay(\Gamma_{i_1} : S_{i_1})$, $Cay(\Gamma_{i_2} : S_{i_2})$, $\cdots$, $Cay(\Gamma_{i_s} : S_{i_s})$. Thus the Cayley graph $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ is connected. □

The following theorem is gotten by the definition of Cayley graph and Theorem 4.4.4.

Theorem 4.4.5 *If* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ *with* $|\Gamma| \geq 3$, *then the Cayley graph* $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$

(1) *is an* $|\widetilde{S}|$*-regular graph;*

(2) *its edge connectivity* $\kappa\left(Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)\right) \geq 2n$.

Proof The assertion (1) is gotten by the definition of $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$. For (2) since every Cayley graph of order≥ 3 is 2-connected, for any two vertices g, h in $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$, there are at least $2n$ edge disjoint paths connecting g and h. Whence, the edge connectivity $\kappa\left(Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)\right) \geq 2n$. □

Applying multi-voltage graphs, we get a structure result for Cayley graphs of a finite multi-group similar to that of Cayley graphs of a finite group.

Theorem 4.4.6 *For a Cayley graph* $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ *of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with* $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, *there is a multi-voltage bouquet* $\varsigma : B_{|\widetilde{S}|} \rightarrow \widetilde{S}$ *such that* $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right) \simeq \left(B_{|\widetilde{S}|}\right)^{\varsigma}$.

Proof Let $\widetilde{S} = \left\{s_i; 1 \leq i \leq \left|\widetilde{S}\right|\right\}$ and $E\left(B_{|\widetilde{S}|}\right) = \left\{L_i; 1 \leq i \leq \left|\widetilde{S}\right|\right\}$. Define a multi-voltage graph on a bouquet $B_{|\widetilde{S}|}$ by

$$\varsigma : L_i \to s_i, \ \ 1 \leq i \leq \left|\widetilde{S}\right|.$$

Then we know that there is an isomorphism τ between $\left(B_{|\widetilde{S}|}\right)^{\varsigma}$ and $Cay\left(\widetilde{\Gamma} : \widetilde{S}\right)$ by defining $\tau(O_g) = g$ for $\forall g \in \widetilde{\Gamma}$, where $V(B_{|\widetilde{S}|}) = \{O\}$. □

Corollary 4.4.6 *For a Cayley graph* $Cay(\Gamma : S)$ *of a finite group* Γ*, there exists a voltage bouquet* $\alpha : B_{|S|} \to S$ *such that* $Cay(\Gamma : S) \simeq (B_{|S|})^{\alpha}$.

§4.5 RESEARCH PROBLEMS

4.5.1 As an efficient way for finding regular covering spaces of a graph, voltage graphs have been gotten more attentions in the past half-century by mathematicians. Unless elementary results on voltage graphs discussed in this chapter, further works for regular covering spaces of graphs can be found in [GrT1], particularly, for finding genus of graphs with more symmetries on surfaces. However, few works can be found in publication for irregular covering spaces of graphs. These multi-voltage graph of type I or type II with multi-groups defined in Sections 4.2-4.3 are candidate for further research on irregular covering spaces of graphs.

Problem 4.5.1 *Applying multi-voltage graphs to get the genus of a graph with less symmetries.*

Problem 4.5.2 *Find new actions of a multi-group on graph, such as the left subaction and its contribution to topological graph theory. What can we say for automorphisms of the lifting of a multi-voltage graph?*

There is a famous conjecture for Cayley graphs of a finite group in algebraic graph theory, i.e., *every connected Cayley graph of order* ≥ 3 *is hamiltonian*. Similarly, we can also present a conjecture for Cayley graphs of a multi-group.

Conjecture 4.5.1 *Every Cayley graph of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with order* ≥ 3 *and* $\left|\bigcap\limits_{i=1}^{n} \Gamma_i\right| \geq 2$ *is hamiltonian.*

4.5.2 As pointed out in [Mao10], for applying combinatorics to other sciences, a good idea is pullback measures on combinatorial objects, initially ignored by the classical combinatorics and reconstructed or make a combinatorial generalization for the classical mathematics. Thus is the CC conjecture following.

Conjecture 4.5.1(CC Conjecture) *The mathematical science can be reconstructed from or made by combinatorialization.*

Remark 4.5.1 We need some further clarifications for this conjecture.

(1) This conjecture assumes that one can select finite combinatorial rulers and axioms to reconstruct or make generalization for classical mathematics.

(2) The classical mathematics is a particular case in the combinatorialization of mathematics, i.e., the later is a combinatorial generalization of the former.

(3) We can make one combinatorialization of different branches in mathematics and find new theorems after then.

More discussions on CC conjecture can be found in references [Mao19] [Mao37]-[Mao38].

4.5.3 The central idea in Section 4.4 is that a graph is equivalent to multi-spaces. Applying infinite graph theory (see [Tho1] for details), we can also define infinite graphs for infinite multi-spaces similar to that Definition 4.4.3.

Problem 4.5.3 *Find the structural properties of infinite graphs of infinite multi-spaces.*

CHAPTER 5.

Multi-Embeddings of Graphs

A geometrical graph G is in fact the graph phase of G. Besides to find combinatorial properties of graphs, a more important thing is to find the behaviors of graphs in spaces, i.e., embedding a graph in space to get its geometrical graph. In last century, many mathematicians concentrated their attention to embedding graphs on surfaces. They have gotten many characteristics of surfaces by combinatorics. Such a way can be also applied to a general space for finding combinatorial behaviors of spaces. Whence, we consider graphs in spaces in this chapter. For this objective, we introduce topological spaces in Section 5.1, multi-surface embeddings, particularly, multi-sphere embedding of graphs with empty overlapping and including multi-embedding on sphere are characterized in Section 5.2 and 2-cell embeddings of graphs on surface in Section 5.3. A general discussion on multi-surface embeddings of graphs and a classification on manifold graphs with enumeration can be found in Section 5.4. Section 5.5 concentrates on the behavior of geometrical graphs, i.e., graph phases in spaces with transformations. All of these materials show how to generalize a classical problem in mathematics by multi-spaces.

§5.1 SURFACES

5.1.1 Topological Space. Let $\mathscr{T}$ be a set. A *topology* on a set $\mathscr{T}$ is a collection $\mathscr{C}$ of subsets of $\mathscr{T}$, called *open sets* satisfying properties following:

(T1) $\emptyset \in \mathscr{C}$ and $\mathscr{T} \in \mathscr{C}$;

(T2) if $U_1, U_2 \in \mathscr{C}$, then $U_1 \cap U_2 \in \mathscr{C}$;

(T3) the union of any collection of open sets is open.

For example, let $\mathscr{T} = \{a, b, c\}$ and $\mathscr{C} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \mathscr{T}\}$. Then $\mathscr{C}$ is a topology on $\mathscr{T}$. Usually, such a topology on a discrete set is called a *discrete topology*, otherwise, a *continuous topology*. A pair $(\mathscr{T}, \mathscr{C})$ consisting of a set $\mathscr{T}$ and a topology $\mathscr{C}$ on $\mathscr{T}$ is called a *topological space* and each element in $\mathscr{T}$ is called a *point* of $\mathscr{T}$. Usually, we also use $\mathscr{T}$ to indicate a topological space if its topology is clear in the context. For example, the Euclidean space $\mathbf{R}^n$ for an integer $n \geq 1$ is a topological space.

For a point u in a topological space $\mathscr{T}$, its an *open neighborhood* is an open set U such that $u \in U$ in $\mathscr{T}$ and a *neighborhood* in $\mathscr{T}$ is a set containing some of its open neighborhoods. Similarly, for a subset A of $\mathscr{T}$, a set U is an *open neighborhood* or *neighborhood* of A if U is open itself or a set containing some open neighborhoods of that set in $\mathscr{T}$. A *basis* in $\mathscr{T}$ is a collection $\mathscr{B}$ of subsets of $\mathscr{T}$ such that $\mathscr{T} = \cup_{B \in \mathscr{B}} B$ and $B_1, B_2 \in \mathscr{B}, x \in B_1 \cap B_2$ implies that $\exists B_3 \in \mathscr{B}$ with $x \in B_3 \subset B_1 \cap B_2$ hold.

Let $\mathscr{T}$ be a topological space and $I = [0, 1] \subset \mathbf{R}$. An *arc a* in $\mathscr{T}$ is defined to be a continuous mapping $a : I \to \mathscr{T}$. We call $a(0)$, $a(1)$ the initial point and end point of a, respectively. A topological space $\mathscr{T}$ is *connected* if there are no open subspaces A and B such that $S = A \cup B$ with $A, B \neq \emptyset$ and called *arcwise-connected* if every two points u, v in $\mathscr{T}$ can be joined by an arc a in $\mathscr{T}$, i.e., $a(0) = u$ and $a(1) = v$. An arc $a : I \to \mathscr{T}$ is a *loop* based at p if $a(0) = a(1) = p \in \mathscr{T}$. A —it degenerated loop $\mathbf{e}_x : I \to x \in S$, i.e., mapping each element in I to a point x, usually called a *point loop*.

A topological space $\mathscr{T}$ is called *Hausdorff* if each two distinct points have disjoint neighborhoods and *first countable* if for each $p \in \mathscr{T}$ there is a sequence $\{U_n\}$ of neighborhoods of p such that for any neighborhood U of p, there is an n such that $U_n \subset U$. The topology is called *second countable* if it has a countable basis.

Let $\{x_n\}$ be a point sequence in a topological space $\mathscr{T}$. If there is a point $x \in \mathscr{T}$ such that for every neighborhood U of u, there is an integer N such that $n \geq N$ implies $x_n \in U$, then $\{u_n\}$ is said *converges* to u or u is a *limit point* of $\{u_n\}$ in the topological space $\mathscr{T}$.

5.1.2 Continuous Mapping. For two topological spaces $\mathscr{T}_1$ and $\mathscr{T}_2$ and a point $u \in \mathscr{T}_1$, a mapping $\varphi : \mathscr{T}_1 \rightarrow \mathscr{T}_2$ is called *continuous at u* if for every neighborhood V of $\varphi(u)$, there is a neighborhood U of u such that $\varphi(U) \subset V$. Furthermore, if φ is continuous at each point u in $\mathscr{T}_1$, then φ is called a *continuous mapping* on $\mathscr{T}_1$.

For examples, the polynomial function $f : \mathbf{R} \rightarrow \mathbf{R}$ determined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and the linear mapping $L : \mathbf{R}^n \rightarrow \mathbf{R}^n$ for an integer $n \geq 1$ are continuous mapping. The following result presents properties of continuous mapping.

Theorem 5.1.1 *Let $\mathscr{R}$, $\mathscr{S}$ and $\mathscr{T}$ be topological spaces. Then*

(1) *A constant mapping $c : \mathscr{R} \rightarrow \mathscr{S}$ is continuous;*

(2) *The identity mapping $Id : \mathscr{R} \rightarrow \mathscr{R}$ is continuous;*

(3) *If $f : \mathscr{R} \rightarrow \mathscr{S}$ is continuous, then so is the restriction $f|_U$ of f to an open subset U of $\mathscr{R}$;*

(4) *If $f : \mathscr{R} \rightarrow \mathscr{S}$ and $g : \mathscr{S} \rightarrow \mathscr{T}$ are continuous at $x \in \mathscr{R}$ and $f(x) \in \mathscr{S}$, then so is their composition mapping $gf : \mathscr{R} \rightarrow \mathscr{T}$ at x.*

Proof The results of (1)-(3) is clear by definition. For (4), notice that f and g are respective continuous at $x \in R$ and $f(x) \in \mathscr{S}$. For any open neighborhood W of point $g(f(x)) \in \mathscr{T}$, $g^{-1}(W)$ is opened neighborhood of $f(x)$ in $\mathscr{S}$. Whence, $f^{-1}(g^{-1}(W))$ is an opened neighborhood of x in $\mathscr{R}$ by definition. Therefore, $g(f)$ is continuous at x. □

A refinement of Theorem 5.1.1(3) enables us to know the following criterion for continuity of a mapping.

Theorem 5.1.2 *Let $\mathscr{R}$ and $\mathscr{S}$ be topological spaces. Then a mapping $f : \mathscr{R} \rightarrow \mathscr{S}$ is continuous if and only if each point of $\mathscr{R}$ has a neighborhood on which f is continuous.*

Proof By Theorem 5.1.1(3), we only need to prove the sufficiency of condition. Let $f : \mathscr{R} \rightarrow \mathscr{S}$ be continuous in a neighborhood of each point of $\mathscr{R}$ and $U \subset \mathscr{S}$. We show that $f^{-1}(U)$ is open. In fact, any point $x \in f^{-1}(U)$ has a neighborhood $V(x)$ on which f is continuous by assumption. The continuity of $f|_{V(x)}$ implies that $(f|_{V(x)})^{-1}(U)$ is open in $V(x)$. Whence it is also open in $\mathscr{R}$. By definition, we are easily find that

$$(f|_{V(x)})^{-1}(U) = \{x \in \mathscr{R} | f(x) \in U\} = f^{-1}(U) \bigcap V(x),$$

in $f^{-1}(U)$ and contains x. Notice that $f^{-1}(U)$ is a union of all such open sets as x ranges over $f^{-1}(U)$. Thus $f^{-1}(U)$ is open followed by this fact. □

For constructing continuous mapping on a union of topological spaces $\mathscr{X}$, the following result is a very useful tool, called the *Gluing Lemma*.

Theorem 5.1.3 *Assume that a topological space $\mathscr{X}$ is a finite union of closed subsets: $\mathscr{X} = \bigcup\limits_{i=1}^{n} X_i$. If for some topological space $\mathscr{Y}$, there are continuous maps $f_i : X_i \to \mathscr{Y}$ that agree on overlaps, i.e., $f_i|_{X_i \bigcap X_j} = f_j|_{X_i \bigcap X_j}$ for all i, j, then there exists a unique continuous $f : \mathscr{X} \to \mathscr{Y}$ with $f|_{X_i} = f_i$ for all i.*

Proof Obviously, the mapping f defined by

$$f(x) = f_i(x), \quad x \in X_i$$

is the unique well defined mapping from $\mathscr{X}$ to $\mathscr{Y}$ with restrictions $f|_{X_i} = f_i$ hold for all i. So we only need to establish the continuity of f on $\mathscr{X}$. In fact, if U is an open set in $\mathscr{Y}$, then

$$\begin{aligned} f^{-1}(U) &= X \bigcap f^{-1}(U) = \left(\bigcup_{i=1}^{n} X_i\right) \bigcap f^{-1}(U) \\ &= \bigcup_{i=1}^{n}\left(X_i \bigcap f^{-1}(U)\right) = \bigcup_{i=1}^{n}\left(X_i \bigcap f_i^{-1}(U)\right) = \bigcup_{i=1}^{n} f_i^{-1}(U). \end{aligned}$$

By assumption, each f_i is continuous. We know that $f_i^{-1}(U)$ is open in X_i. Whence, $f^{-1}(U)$ is open in $\mathscr{X}$. Thus f is continuous on $\mathscr{X}$. □

Let $\mathscr{X}$ be a topological space. A collection $C \subset \mathscr{P}(\mathscr{X})$ is called to be a *cover* of $\mathscr{X}$ if

$$\bigcup_{C \in C} C = \mathscr{X}.$$

If each set in C is open, then C is called an *opened cover* and if $|C|$ is finite, it is called a *finite cover* of $\mathscr{X}$. A topological space is *compact* if there exists a finite cover in its any opened cover and *locally compact* if it is Hausdorff with a compact neighborhood for its each point. As a consequence of Theorem 5.1.3, we can apply the gluing lemma to ascertain continuous mappings shown in the next.

Corollary 5.1.1 *Let Let $\mathscr{X}$ and $\mathscr{Y}$ be topological spaces and $\{A_1, A_2, \cdots, A_n\}$ be a finite opened cover of a topological space $\mathscr{X}$. If a mapping $f : \mathscr{X} \to \mathscr{Y}$ is continuous constrained on each A_i, $1 \le i \le n$, then f is a continuous mapping.*

5.1.3 Homeomorphic Space. Let $\mathscr{S}$ and $\mathscr{T}$ be two topological spaces. They are *homeomorphic* if there is a $1-1$ continuous mapping $\varphi : \mathscr{S} \to \mathscr{T}$ such that the inverse maping $\varphi^{-1} : \mathscr{T} \to \mathscr{S}$ is also continuous. Such a mapping φ is called a *homeomorphic* or *topological* mapping. A few examples of homeomorphic spaces can be found in the following.

Example 5.1.1 Each of the following topological space pairs are homeomorphic.

(1) A Euclidean space $\mathbf{R}^n$ and an opened unit n-ball $B^n = \{ (x_1, x_2, \cdots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1 \}$;

(2) A Euclidean plane $\mathbf{R}^{n+1}$ and a unit sphere $S^n = \{ (x_1, x_2, \cdots, x_{n+1}) \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \}$ with one point $p = (0, 0, \cdots, 0, 1)$ on it removed.

In fact, define a mapping f from B^n to $\mathbf{R}^n$ for (1) by

$$f(x_1, x_2, \cdots, x_n) = \frac{(x_1, x_2, \cdots, x_n)}{1 - \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

for $\forall (x_1, x_2, \cdots, x_n) \in B^n$. Then its inverse is

$$f^{-1}(x_1, x_2, \cdots, x_n) = \frac{(x_1, x_2, \cdots, x_n)}{1 + \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

for $\forall (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$. Clearly, both f and f^{-1} are continuous. So B^n is homeomorphic to $\mathbf{R}^n$. For (2), define a mapping f from $S^n - p$ to $\mathbf{R}^{n+1}$ by

$$f(x_1, x_2, \cdots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \cdots, x_n).$$

Its inverse $f^{-1} : \mathbf{R}^{n+1} \to S^n - p$ is determined by

$$f^{-1}(x_1, x_2, \cdots, x_{n+1}) = (t(x)x_1, \cdots, t(x)x_n, 1 - t(x)),$$

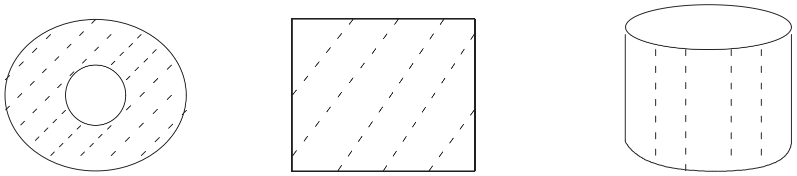
where

$$t(x) = \frac{2}{1 + x_1^2 + x_2^2 + \cdots + x_{n+1}^2}.$$

Notice that both f and f^{-1} are continuous. Thus $S^n - p$ is homeomorphic to $\mathbf{R}^{n+1}$.

5.1.4 Surface. For an integer $n \geq 1$, an *n-dimensional topological manifold* is a second countable Hausdorff space such that each point has an open neighborhood homeomorphic to an open n-dimensional ball $B^n = \{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$ in $\mathbf{R}^n$. We assume all manifolds is connected considered in this book. A 2-manifold is usually called *surface* in literature. Several examples of surfaces are shown in the following.

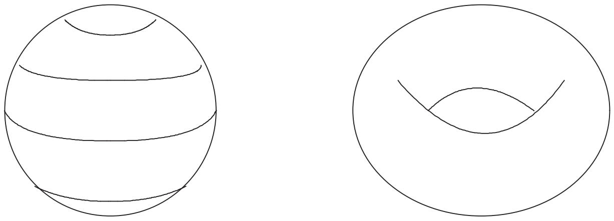
Example 5.1.1 These 2-manifolds shown in the Fig.5.1.1 are surfaces with boundary.



plane torus rectangle cylinder

Fig.5.1.1

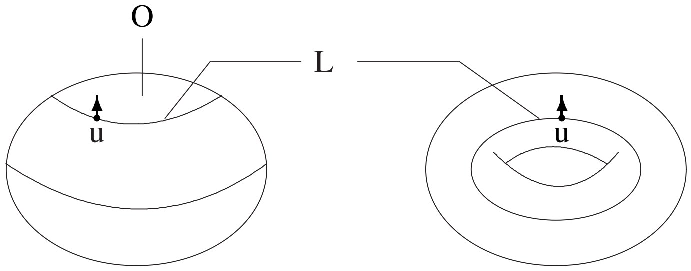
Example 5.1.2 These 2-manifolds shown in the Fig.5.1.2 are surfaces without boundary.



sphere torus

Fig.5.1.2

By definition, we can always distinguish the right-side and left-side when one object moves along an arc on a surface S. Now let **N** be a unit normal vector of the surface S. Consider the result of a normal vector moves along a loop L on surfaces in Fig.5.1.1 and Fig.5.1.2. We find the direction of **N** is unchanged as it come back at the original point u. For example, it moves on the sphere and torus shown in the Fig.5.1.3 following.



sphere torus

Fig.5.1.3

Such loops L in Fig.5.1.3 are called *orientation-preserving*. However, there are also loops L in surfaces which are not orientation-preserving. In such case, we get the opposite direction of **N** as it come back at the original point v. Such a loop is called *orientation-reversing*. For example, the process (1)-(3) for getting the famous Möbius strip shown in Fig.5.1.4, in where the loop L is an orientation-reversing loop.

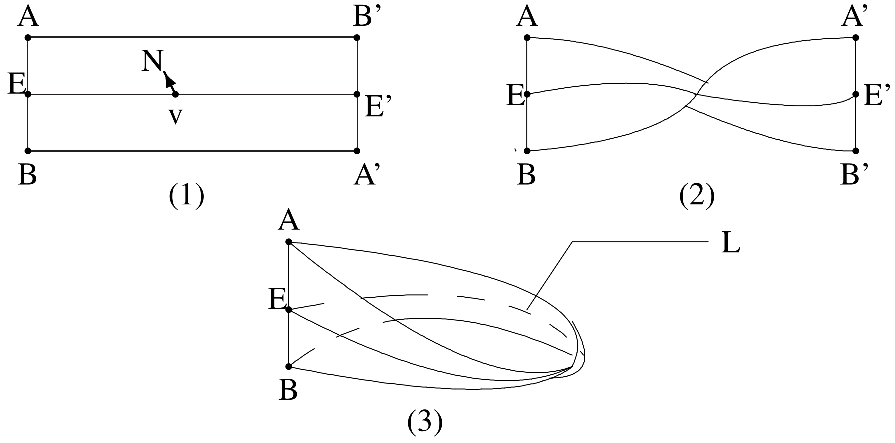


Fig.4.1.4

A surface S is defined to be *orientable* if every loop on S is orientation-preserving. Otherwise, *non-orientable* if there at least one orientation-reversing loop on S. Whence, the surfaces in Examples 5.1.1-5.1.2 are orientable and the Möbius strip are non-orientable. It should be noted that the boundary of a Möbius strip is a closed arc formed by AB' and $A'B$. Gluing the boundary of a Möbius strip by a 2-dimensional ball B^2, we get a non-orientable surface without boundary, which is usually called *crosscap* in literature.

§5.2 GRAPHS IN SPACES

5.2.1 Graph Embedding. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two topological spaces. An embedding of $\mathcal{E}_1$ in $\mathcal{E}_2$ is a one-to-one continuous mapping $f : \mathcal{E}_1 \to \mathcal{E}_2$. Certainly, the same problem can be also considered for $\mathcal{E}_2$ being a metric space. By topological view, a graph is nothing but a 1-complex, we consider the embedding problem for graphs in spaces. The same problem had been considered by Grümbaum in [Gru1]-[Gru3] for graphs in spaces, and references [GrT1], [Liu1]-[Liu4], [MoT1] and [Whi1] for graphs on surfaces.

5.2.2 Graph in Manifold. Let G be a connected graph. For $\forall v \in V(G)$, a *space permutation* $P(v)$ of v is a permutation on $N_G(v) = \{u_1, u_2, \cdots, u_{\rho_G(v)}\}$ and all space permutation of a vertex v is denoted by $\mathcal{P}_s(v)$. A *space permutation* $P_s(G)$ *of a graph* G is defined to be

$$P_s(G) = \{P(v)|\forall v \in V(G), P(v) \in \mathcal{P}_s(v)\}$$

and a *permutation system* $\mathcal{P}_s(G)$ *of* G to be all space permutation of G. Then we know the following characteristic for an embedded graph in an n-manifold $\mathbf{M^n}$ with $n \geq 3$.

Theorem 5.2.1 *For an integer* $n \geq 3$*, every space permutation* $P_s(G)$ *of a graph* G *defines a unique embedding of* $G \to \mathbf{M}^n$. *Conversely, every embedding of a graph* $G \to \mathbf{M}^n$ *defines a space permutation of* G.

Proof Assume G is embedded in an n-manifold $\mathbf{M}^n$. For $\forall v \in V(G)$, define an $(n-1)$-ball $B^{n-1}(v)$ to be $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ with center at v and radius r as small as needed. Notice that all auto-homeomorphisms $\mathrm{Aut}B^{n-1}(v)$ of $B^{n-1}(v)$ is a group under the composition operation and two points $A = (x_1, x_2, \cdots, x_n)$ and $B = (y_1, y_2, \cdots, y_n)$ in $B^{n-1}(v)$ are said to be combinatorially equivalent if there exists an auto-homeomorphism $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma(A) = B$. Consider intersection points of edges in $E_G(v, N_G(v))$ with $B^{n-1}(v)$. We get a permutation $P(v)$ on these points, or equivalently on $N_G(v)$ by $(A, B, \cdots, C, D)$ being a cycle of $P(v)$ if and only if there exists $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma^i(A) = B, \cdots, \varsigma^j(C) = D$ and $\varsigma^l(D) = A$, where $i, \cdots, j, l$ are integers. Thereby we get a space permutation $P_s(G)$ of G.

Conversely, for a space permutation $P_s(G)$, we can embed G in $\mathbf{M}^n$ by embedding each vertex $v \in V(G)$ to a point X of $\mathbf{M}^n$ and arranging vertices in one cycle of $P_s(G)$ of $N_G(v)$ as the same orbit of $\langle\sigma\rangle$ action on points of $N_G(v)$ for $\sigma \in \mathrm{Aut}B^{n-1}(X)$. Whence we get an embedding of G in the manifold $\mathbf{M}^n$. □

Theorem 5.2.1 establishes a relation for an embedded graph in an n-dimensional manifold with a permutation, which enables one combinatorially defining graphs embedded in n-dimensional manifolds.

Corollary 5.2.1 *For a graph* G*, the number of embeddings of* G *in* $\mathbf{M}^n, n \geq 3$ *is*

$$\prod_{v \in V(G)} \rho_G(v)!.$$

For applying graphs in spaces to theoretical physics, we consider an embedding of graph in an manifold with additional conditions, which enables us to find good behavior of a graph in spaces. On the first, we consider the rectilinear embeddings of graphs in a Euclid space.

Definition 5.2.1 *For a given graph G and a Euclid space* $\mathbf{E}$*, a rectilinear embedding of G in* $\mathbf{E}$ *is a one-to-one continuous mapping $\pi : G \to \mathbf{E}$ such that*

(1) *For $\forall e \in E(G)$, $\pi(e)$ is a segment of a straight line in* $\mathbf{E}$*;*

(2) *For any two edges $e_1 = (u, v), e_2 = (x, y)$ in $E(G)$, $(\pi(e_1) \setminus \{\pi(u), \pi(v)\}) \bigcap (\pi(e_2) \setminus \{\pi(x), \pi(y)\}) = \emptyset$.*

In $\mathbf{R}^3$, a rectilinear embedding of K_4 and a cube Q_3 are shown in Fig.5.2.1 following.

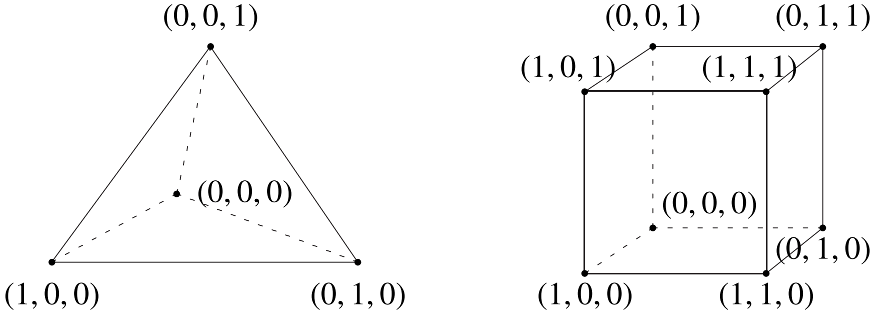


Fig 5.2.1

In general, we know the following result for rectilinear embedding of graphs G in Euclid space $\mathbf{R}^n, n \geq 3$.

Theorem 5.2.2 *For any simple graph G of order n, there is a rectilinear embedding of G in $\mathbf{R}^n$ with $n \geq 3$.*

Proof Notice that this assertion is true for any integer $n \geq 3$ if it is hold for $n = 3$. In $\mathbf{R}^3$, choose n points $(t_1, t_1^2, t_1^3), (t_2, t_2^2, t_2^3), \cdots, (t_n, t_n^2, t_n^3)$, where $t_1, t_2, \cdots, t_n$ are n different real numbers. For integers $i, j, k, l, 1 \leq i, j, k, l \leq n$, if a straight line passing through vertices (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) intersects with a straight line passing through vertices (t_k, t_k^2, t_k^3) and (t_l, t_l^2, t_l^3), then there must be

$$\begin{vmatrix} t_k - t_i & t_j - t_i & t_l - t_k \\ t_k^2 - t_i^2 & t_j^2 - t_i^2 & t_l^2 - t_k^2 \\ t_k^3 - t_i^3 & t_j^3 - t_i^3 & t_l^3 - t_k^3 \end{vmatrix} = 0,$$

which implies that there exist integers $s, f \in \{k, l, i, j\}$, $s \neq f$ such that $t_s = t_f$, a contradiction.

Now let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We embed the graph G in $\mathbf{R}^3$ by a mapping $\pi : G \to \mathbf{R}^3$ with $\pi(v_i) = (t_i, t_i^2, t_i^3)$ for $1 \leq i \leq n$ and if $v_iv_j \in E(G)$, define $\pi(v_iv_j)$ being the segment between points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) of a straight line passing through points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3). Then π is a rectilinear embedding of the graph G in $\mathbf{R}^3$. $\square$

5.2.3 Multi-Surface Embedding. For a graph G and a surface S, an *immersion* ι of G on S is a one-to-one continuous mapping $\iota : G \to S$ such that for $\forall e \in E(G)$, if $e = (u, v)$, then $\iota(e)$ is a curve connecting $\iota(u)$ and $\iota(v)$ on S. The following two definitions are generalization of embedding of graph on surface.

Definition 5.2.2 *Let G be a graph and S a surface in a metric space $\mathcal{E}$. A pseudo-embedding of G on S is a one-to-one continuous mapping $\pi : G \to \mathcal{E}$ such that there exists vertices $V_1 \subset V(G)$, $\pi|_{\langle V_1 \rangle}$ is an immersion on S with each component of $S \setminus \pi(\langle V_1 \rangle)$ isomorphic to an open 2-disk.*

Definition 5.2.3 *Let G be a graph with a vertex set partition $V(G) = \bigcup\limits_{j=1}^{k} V_i$, $V_i \bigcap V_j = \emptyset$ for $1 \leq i, j \leq k$ and let $S_1, S_2, \cdots, S_k$ be surfaces in a metric space $\mathcal{E}$ with $k \geq 1$. A multi-embedding of G on $S_1, S_2, \cdots, S_k$ is a one-to-one continuous mapping $\pi : G \to \mathcal{E}$ such that for any integer i, $1 \leq i \leq k$, $\pi|_{\langle V_i \rangle}$ is an immersion with each component of $S_i \setminus \pi(\langle V_i \rangle)$ isomorphic to an open 2-disk.*

Notice that if $\pi(G) \bigcap (S_1 \bigcup S_2 \cdots \bigcup S_k) = \pi(V(G))$, then every $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G. We say it to be a *trivial multi-embedding* of G on $S_1, S_2, \cdots, S_k$. If $k = 1$, then every trivial multi-embedding is a trivial pseudo-embedding of G on S_1. The main object of this section is to find nontrivial multi-embedding of G on $S_1, S_2, \cdots, S_k$ with $k \geq 1$. The existence pseudo-embedding of a graph G is obvious by definition. We concentrate our attention on characteristics of multi-embeddings of a graph.

For a graph G, let $G_1, G_2, \cdots, G_k$ be all vertex-induced subgraphs of G. For any integers i, j, $1 \leq i, j \leq k$, if $V(G_i) \bigcap V(G_j) = \emptyset$, such a set consisting of subgraphs $G_1, G_2, \cdots, G_k$ are called a *block decomposition* of G and denoted by $G = \biguplus\limits_{i=1}^{k} G_i$. The *planar block number* $n_p(G)$ of G is defined by

$$n_p(G) = \min\left\{ k | G = \biguplus_{i=1}^{k} G_i, \text{for any integer } i, 1 \leq i \leq k, G_i \text{ is planar} \right\}.$$

Then we get a result for the planar black number of a graph G in the following.

Theorem 5.2.3 *A graph G has a nontrivial multi-embedding on s spheres $P_1, P_2, \cdots, P_s$ with empty overlapping if and only if $n_p(G) \leq s \leq |G|$.*

Proof Assume G has a nontrivial multi-embedding on spheres $P_1, P_2, \cdots, P_s$. Since $|V(G) \bigcap P_i| \geq 1$ for any integer $i, 1 \leq i \leq s$, we know that

$$|G| = \sum_{i=1}^{s} \left| V(G) \bigcap P_i \right| \geq s.$$

By definition, if $\pi : G \rightarrow \mathbf{R}^3$ is a nontrivial multi-embedding of G on $P_1, P_2, \cdots, P_s$, then for any integer $i, 1 \leq i \leq s$, $\pi^{-1}(P_i)$ is a planar induced graph. Therefore,

$$G = \biguplus_{i=1}^{s} \pi^{-1}(P_i),$$

and we get that $s \geq n_p(G)$.

Now if $n_p(G) \leq s \leq |G|$, there is a block decomposition $G = \biguplus_{i=1}^{s} G_s$ of G such that G_i is a planar graph for any integer $i, 1 \leq i \leq s$. Whence we can take s spheres $P_1, P_2, \cdots, P_s$ and define an embedding $\pi_i : G_i \rightarrow P_i$ of G_i on sphere P_i for any integer $i, 1 \leq i \leq s$. Define an immersion $\pi : G \rightarrow \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = \left(\bigcup_{i=1}^{s} \pi(G_i) \right) \bigcup \left\{ (v_i, v_j) | v_i \in V(G_i), v_j \in V(G_j), (v_i, v_j) \in E(G), 1 \leq i, j \leq s \right\}.$$

Then $\pi : G \rightarrow \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. □

For example, a multi-embedding of K_6 on two spheres is shown in Fig.5.2.2, where, $\langle \{x, y, z\} \rangle$ is on one sphere S_1 and $\langle \{u, v, w\} \rangle$ on another S_2.

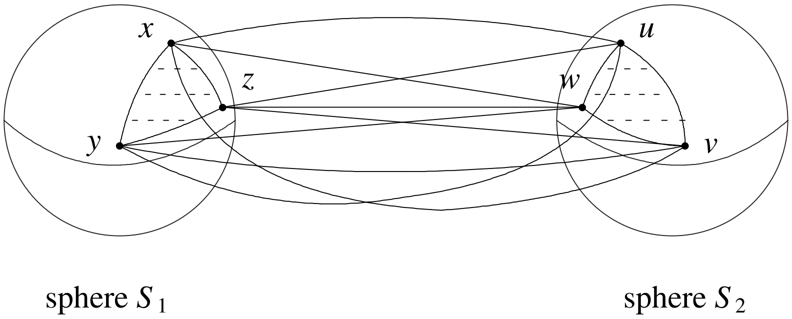


Fig 5.2.2

For a complete or a complete bipartite graph, the number $n_p(G)$ is determined in the following result.

Theorem 5.2.4 *For any integers* $n,\ m \geq 1$, *the numbers* $n_p(K_n)$ *and* $n_p(K(m,n))$ *are respectively*

$$n_p(K_n) = \left\lceil \frac{n}{4} \right\rceil \ \ \textit{and}\ \ n_p(K(m,n)) = 2,$$

if $m \geq 3, n \geq 3$, *otherwise* 1, *respectively.*

Proof Notice that every vertex-induced subgraph of a complete graph K_n is also a complete graph. By Theorem 2.1.16, we know that K_5 is non-planar. Thereby we get that

$$n_p(K_n) = \left\lceil \frac{n}{4} \right\rceil$$

by definition of $n_p(K_n)$. Now for a complete bipartite graph K(m,n), any vertex-induced subgraph by choosing s and l vertices from its two partite vertex sets is still a complete bipartite graph. According to Theorem 2.2.5, $K(3,3)$ is non-planar and $K(2,k)$ is planar. If $m \leq 2$ or $n \leq 2$, we get that $n_p(K(m,n)) = 1$. Otherwise, $K(m,n)$ is non-planar. Thereby we know that $n_p(K(m,n)) \geq 2$.

Let $V(K(m,n)) = V_1 \bigcup V_2$, where V_1, V_2 are its partite vertex sets. If $m \geq 3$ and $n \geq 3$, we choose vertices $u,v \in V_1$ and $x,y \in V_2$. Then the vertex-induced subgraphs $\langle \{u,v\} \bigcup V_2 \setminus \{x,y\} \rangle$ and $\langle \{x,y\} \bigcup V_2 \setminus \{u,v\} \rangle$ in $K(m,n)$ are planar graphs. Whence, $n_p(K(m,n)) = 2$ by definition. □

The position of surfaces $S_1, S_2, \cdots, S_k$ in a topological space $\mathcal{E}$ also influences the existence of multi-embeddings of a graph. Among these cases, an interesting case is there exists an arrangement $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$ for $S_1, S_2, \cdots, S_k$ such that in $\mathcal{E}$, S_{i_j} is a subspace of $S_{i_{j+1}}$ for any integer $j, 1 \leq j \leq k$. In this case, the multi-embedding is called an *including multi-embedding* of G on surfaces $S_1, S_2, \cdots, S_k$.

Theorem 5.2.5 *A graph* G *has a nontrivial including multi-embedding on spheres* $P_1 \supset P_2 \supset \cdots \supset P_s$ *if and only if there is a block decomposition* $G = \biguplus\limits_{i=1}^{s} G_i$ *of* G *such that for any integer* $i, 1 < i < s$,

(1) G_i *is planar;*

(2) *for* $\forall v \in V(G_i)$, $N_G(x) \subseteq \left(\bigcup\limits_{j=i-1}^{i+1} V(G_j) \right)$.

Proof Notice that in the case of spheres, if the radius of a sphere is tending to infinite, an embedding of a graph on this sphere is tending to a planar embedding. From this observation, we get the necessity of these conditions.

Now if there is a block decomposition $G = \biguplus_{i=1}^{s} G_i$ of G such that G_i is planar for any integer $i, 1 < i < s$ and $N_G(x) \subseteq \left(\bigcup_{j=i-1}^{i+1} V(G_j)\right)$ for $\forall v \in V(G_i)$, we can so place s spheres $P_1, P_2, \cdots, P_s$ in $\mathbf{R}^3$ that $P_1 \supset P_2 \supset \cdots \supset P_s$. For any integer $i, 1 \leq i \leq s$, we define an embedding $\pi_i : G_i \to P_i$ of G_i on sphere P_i.

Since $N_G(x) \subseteq \left(\bigcup_{j=i-1}^{i+1} V(G_j)\right)$ for $\forall v \in V(G_i)$, define an immersion $\pi : G \to \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = \left(\bigcup_{i=1}^{s} \pi(G_i)\right) \bigcup \left\{(v_i, v_j) | j = i-1, i, i+1 \ for \ 1 < i < s \ and \ (v_i, v_j) \in E(G)\right\}.$$

Then $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. □

Corollary 5.2.2 *If a graph G has a nontrivial including multi-embedding on spheres $P_1 \supset P_2 \supset \cdots \supset P_s$, then the diameter $D(G) \geq s - 1$.*

§5.3 GRAPHS ON SURFACES

5.3.1 2-Cell Embedding. For a graph $G = (V(G), E(G), I(G))$ and a surface S, an embedding of G on S is the case of $k = 1$ in Definition 5.2.3, which is also an embedding of graph in a 2-manifold. It can be shown immediately that if there exists an embedding of G on S, then G is connected. Otherwise, we can get a component in $S \setminus \pi(G)$ not isomorphic to an open 2-disk. Thus all graphs considered in this subsection are connected.

Let G be a graph. For $v \in V(G)$, denote all of edges incident with the vertex v by $N_G^e(v) = \{e_1, e_2, \cdots, e_{\rho_G(v)}\}$. A permutation $C(v)$ on $e_1, e_2, \cdots, e_{\rho_G(v)}$ is said to be a *pure rotation* of v. All such pure rotations incident with a vertex v is denoted by $\varrho(v)$. A *pure rotation system* of G is defined by

$$\rho(G) = \{C(v) | C(v) \in \varrho(v) \text{ for } \forall v \in V(G)\}$$

and all pure rotation systems of G is denoted by $\varrho(G)$.

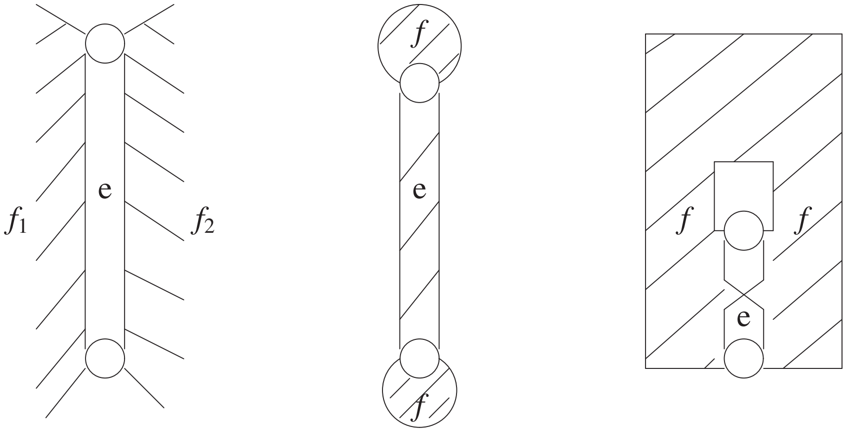
Notice that in the case of embedded graphs on surfaces, a 1-dimensional ball is just a circle. By Theorem 5.2.1, we get a useful characteristic for embedding of graphs on orientable surfaces, first found by Heffter in 1891 and then formulated by Edmonds in 1962. It can be restated as follows.

Theorem 5.3.1 *Every pure rotation system for a graph G induces a unique embedding of G into an orientable surface. Conversely, every embedding of a graph G into an orientable surface induces a unique pure rotation system of G.*

According to this theorem, we know that the number of all embeddings of a graph G on orientable surfaces is $\prod_{v\in V(G)}(\rho_G(v)-1)!$.

By topological view, an embedded vertex or face can be viewed as a disk, and an embedded edge can be viewed as a 1-band which is defined as a topological space B together with a homeomorphism $h : I\times I \rightarrow B$, where $I = [0,1]$, the unit interval. Whence, an edge in an embedded graph has two sides. One side is $h((0,x))$, $x\in I$. Another is $h((1,x))$, $x\in I$.

For an embedded graph G on a surface, the two sides of an edge $e\in E(G)$ may lie in two different faces f_1 and f_2, or in one face f without a twist ,or in one face f with a twist such as those cases (a), or (b), or (c) shown in Fig.5.3.1.



(a) (b) (c)

Fig 5.3.1

Now we define a rotation system $\rho^L(G)$ to be a pair $(\mathcal{J},\lambda)$, where $\mathcal{J}$ is a pure rotation system of G, and $\lambda : E(G)\rightarrow Z_2$. The edge with $\lambda(e)=0$ or $\lambda(e)=1$ is called *type* 0 or *type 1* edge, respectively. The *rotation system* $\varrho^L(G)$ of a graph G are defined by

$$\varrho^L(G)=\{(\mathcal{J},\lambda)|\mathcal{J}\in\varrho(G),\lambda : E(G)\rightarrow Z_2\}.$$

By Theorem 5.2.1 we know the following characteristic for embedding graphs on locally orientable surfaces.

Theorem 5.3.2 *Every rotation system on a graph G defines a unique locally orientable embedding of $G \to S$. Conversely, every embedding of a graph $G \to S$ defines a rotation system for G.*

Notice that in any embedding of a graph G, there exists a spanning tree T such that every edge on this tree is type 0 (See also [GrT1] for details). Whence, the number of all embeddings of a graph G on locally orientable surfaces is

$$2^{\beta(G)} \prod_{v \in V(G)} (\rho_G(v) - 1)!$$

and the number of all embedding of G on non-orientable surfaces is

$$(2^{\beta(G)} - 1) \prod_{v \in V(G)} (\rho(v) - 1)!.$$

The following result is the famous *Euler-Poincaré* formula for embedding a graph on a surface.

Theorem 5.3.3 *If a graph G can be embedded into a surface S, then*

$$\nu(G) - \varepsilon(G) + \phi(G) = \chi(S),$$

where $\nu(G), \varepsilon(G)$ and $\phi(G)$ are the order, size and the number of faces of G on S, and $\chi(S)$ is the Euler characteristic of S, i.e.,

$$\chi(S) = \begin{cases} 2 - 2p, & if\ S\ is\ orientable, \\ 2 - q, & if\ S\ is\ non-orientable. \end{cases}$$

For a given graph G and a surface S, whether G embeddable on S is uncertain. We use the notation $G \to S$ denoting that G can be embeddable on S. Define the *orientable genus range* $GR^O(G)$ and the *non-orientable genus range* $GR^N(G)$ of a graph G by

$$GR^O(G) = \left\{ \frac{2 - \chi(S)}{2} \mid G \to S, S\ is\ an\ orientable\ surface \right\},$$

$$GR^N(G) = \{2 - \chi(S) | G \to S, S\ is\ a\ non-orientable\ surface\},$$

respectively and the orientable or non-orientable genus $\gamma(G), \overline{\gamma}(G)$ by

$$\gamma(G) = \min\left\{p | p \in GR^O(G)\right\}, \ \ \gamma_M(G) = \max\left\{p | p \in GR^O(G)\right\},$$

$$\widetilde{\gamma}(G) = \min\left\{q | q \in GR^N(G)\right\}, \ \ \widetilde{\gamma}_M(G) = \max\left\{q | q \in GR^O(G)\right\}.$$

Theorem 5.3.4 *Let G be a connected graph. Then*

$$GR^O(G) = [\gamma(G), \gamma_M(G)].$$

Proof Notice that if we delete an edge e and its adjacent faces from an embedded graph G on surface S, we get two holes at most, see Fig.25 also. This implies that $|\phi(G) - \phi(G-e)| \leq 1$.

Now assume G has been embedded on a surface of genus $\gamma(G)$ and $V(G) = \{u, v, \cdots, w\}$. Consider those of edges adjacent with u. Not loss of generality, we assume the rotation of G at vertex v is $(e_1, e_2, \cdots, e_{\rho_G(u)})$. Construct an embedded graph sequence $G_1, G_2, \cdots, G_{\rho_G(u)!}$ by

$$\begin{aligned}
&\varrho(G_1) = \varrho(G);\\
&\varrho(G_2) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_1, e_3, \cdots, e_{\rho_G(u)})\};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots;\\
&\varrho(G_{\rho_G(u)-1}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_3, \cdots, e_{\rho_G(u)}, e_1)\};\\
&\varrho(G_{\rho_G(u)}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_3, e_2, \cdots, e_{\rho_G(u)}, e_1)\};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots;\\
&\varrho(G_{\rho_G(u)!}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_{\rho_G(u)}, \cdots, e_2, e_1,)\}.
\end{aligned}$$

For any integer i, $1 \leq i \leq \rho_G(u)!$, since $|\phi(G) - \phi(G-e)| \leq 1$ for $\forall e \in E(G)$, we know that $|\phi(G_{i+1}) - \phi(G_i)| \leq 1$. Whence, $|\chi(G_{i+1}) - \chi(G_i)| \leq 1$.

Continuing the above process for every vertex in G we finally get an embedding of G with the maximum genus $\gamma_M(G)$. Since in this sequence of embeddings of G, the genus of two successive surfaces differs by at most one, thus $GR^O(G) = [\gamma(G), \gamma_M(G)]$. □

The genus problem, i.e., *to determine the minimum orientable or non-orientable genus of a graph* is NP-complete (See [GrT1] for details). Ringel and Youngs got the genus of K_n completely by *current graphs* (a dual form of voltage graphs) as follows.

Theorem 5.3.5 *For a complete graph K_n and a complete bipartite graph $K(m,n)$ with integers $m, n \geq 3$,*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ and } \gamma(K(m,n)) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Outline proofs for $\gamma(K_n)$ in Theorem 2.3.10 can be found in [GrT1], [Liu1] and [MoT1], and a complete proof is contained in [Rin1]. A proof for $\gamma(K(m,n))$ in Theorem 5.3.5 can be also found in [GrT1], [Liu1] and [MoT1].

For the maximum genus $\gamma_M(G)$ of a graph, the time needed for computation is bounded by a polynomial function on the number of $\nu(G)$ ([GrT1]). In 1979, Xuong got the following result.

Theorem 5.3.6 *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q - n + 1) - \frac{1}{2}\min_{T} c_{odd}(G \setminus E(T)),$$

where the minimum is taken over all spanning trees T of G and $c_{odd}(G \setminus E(T))$ denotes the number of components of $G \setminus E(T)$ with an odd number of edges.

In 1981, Nebeský derived another important formula for the maximum genus of a graph. For a connected graph G and $A \subset E(G)$, let $c(A)$ be the number of connected component of $G \setminus A$ and let $b(A)$ be the number of connected components X of $G \setminus A$ such that $|E(X)| \equiv |V(X)|(mod2)$. With these notations, his formula can be restated as in the next theorem.

Theorem 5.3.7 *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q - n + 2) - \max_{A \subseteq E(G)}\{c(A) + b(A) - |A|\}.$$

Corollary 5.3.1 *The maximum genus of K_n and $K(m,n)$ are given by*

$$\gamma_M(K_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor \text{ and } \gamma_M(K(m,n)) = \left\lfloor \frac{(m-1)(n-1)}{2} \right\rfloor,$$

respectively.

Now we turn to non-orientable embedding of a graph G. For $\forall e \in E(G)$, we define an *edge-twisting surgery* $\otimes(e)$ to be given the band of e an extra twist such as that shown in Fig.5.3.2.

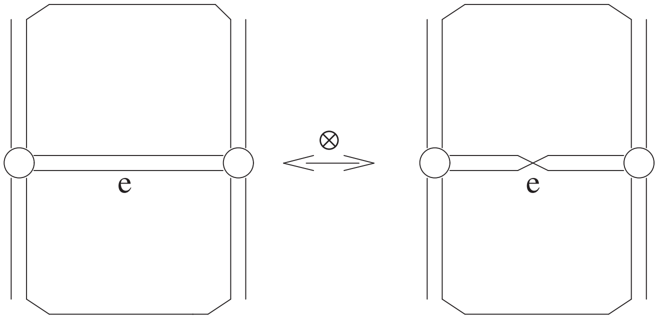


Fig 5.3.2

Notice that for an embedded graph G on a surface S, $e \in E(G)$, if two sides of e are in two different faces, then $\otimes(e)$ will make these faces into one and if two sides of e are in one face, $\otimes(e)$ will divide the one face into two. This property of $\otimes(e)$ enables us to get the following result for the crosscap range of a graph.

Theorem 5.3.8 *Let G be a connected graph. Then*

$$GR^N(G) = [\widetilde{\gamma}(G), \beta(G)],$$

where $\beta(G) = \varepsilon(G) - \nu(G) + 1$ is called the Betti number of G.

Proof It can be checked immediately that $\widetilde{\gamma}(G) = \widetilde{\gamma}_M(G) = 0$ for a tree G. If G is not a tree, we have known there exists a spanning tree T such that every edge on this tree is type 0 for any embedding of G.

Let $E(G) \setminus E(T) = \{e_1, e_2, \cdots, e_{\beta(G)}\}$. Adding the edge e_1 to T, we get a two faces embedding of $T + e_1$. Now make edge-twisting surgery on e_1. Then we get a one face embedding of $T + e_1$ on a surface. If we have get a one face embedding of $T + (e_1 + e_2 + \cdots + e_i)$, $1 \leq i < \beta(G)$, adding the edge e_{i+1} to $T + (e_1 + e_2 + \cdots + e_i)$ and make $\otimes(e_{i+1})$ on the edge e_{i+1}. We also get a one face embedding of $T + (e_1 + e_2 + \cdots + e_{i+1})$ on a surface again.

Continuing this process until all edges in $E(G) \setminus E(T)$ have a twist, we finally get a one face embedding of $T + (E(G) \setminus E(T)) = G$ on a surface. Since the number of twists in each circuit of this embedding of G is $1(mod2)$, this embedding is non-orientable with only one face. By the Euler-Poincaré formula, we know its genus $\widetilde{g}(G)$

$$\widetilde{g}(G) = 2 - (\nu(G) - \varepsilon(G) + 1) = \beta(G).$$

For a minimum non-orientable embedding $\mathcal{E}_G$ of G, i.e., $\widetilde{\gamma}(\mathcal{E}_G) = \widetilde{\gamma}(G)$, one can selects an edge e that lies in two faces of the embedding $\mathcal{E}_G$ and makes $\otimes(e)$. Thus in at most $\widetilde{\gamma}_M(G) - \widetilde{\gamma}(G)$ steps, one has obtained all of embeddings of G on every non-orientable surface N_s with $s \in [\widetilde{\gamma}(G), \widetilde{\gamma}_M(G)]$. Therefore,

$$GR^N(G) = [\widetilde{\gamma}(G), \beta(G)] \qquad \square$$

Corollary 5.3.2 *Let G be a connected graph with p vertices and q edges. Then*

$$\widetilde{\gamma}_M(G) = q - p + 1.$$

Theorem 5.3.9 For a complete graph K_n and a complete bipartite graph $K(m,n)$, $m,n \geq 3$,

$$\widetilde{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

with an exception value $\widetilde{\gamma}(K_7) = 3$ *and*

$$\widetilde{\gamma}(K(m,n)) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

A complete proof of this theorem is contained in [Rin1], Outline proofs of Theorem 5.3.9 can be found in [Liu1].

5.3.2 Combinatorial Map. Geometrically, an embedded graph of G on a surface is called a combinatorial map M and say G underlying M. Tutte [Tut2] found an algebraic representation for an embedded graph on a locally orientable surface in 1973, which transfers a geometrical partition of a surface to a permutation in algebra.

A *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a permutation $\mathcal{P}$ acting on $\mathcal{X}_{\alpha,\beta}$ of a disjoint union of quadricells Kx of $x \in X$, where X is a finite set and $K = \{1, \alpha, \beta, \alpha\beta\}$ is Klein 4-group with conditions following hold:

(1) $\forall x \in \mathcal{X}_{\alpha,\beta}$, there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$;

(2) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(3) The group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

The *vertices* of a combinatorial map are defined to be pairs of conjugate orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$, *edges* to be orbits of K on $\mathcal{X}_{\alpha,\beta}$ and *faces* to be pairs of conjugate orbits of $\mathcal{P}\alpha\beta$ action on $\mathcal{X}_{\alpha,\beta}$. For determining a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is orientable or not, the following condition is needed.

(4) *If the group* $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ *is transitive on* $\mathcal{X}_{\alpha,\beta}$, *then M is non-orientable. Otherwise, orientable.*

For example, the graph $D_{0.4.0}$ (a dipole with 4 multiple edges) on Klein bottle shown in Fig.5.3.3 can be algebraic represented by a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with

$$\mathcal{X}_{\alpha,\beta} = \bigcup_{e \in \{x,y,z,w\}} \{e, \alpha e, \beta e, \alpha\beta e\},$$

$$\mathcal{P} = (x, y, z, w)(\alpha\beta x, \alpha\beta y, \beta z, \beta w)(\alpha x, \alpha w, \alpha z, \alpha y)(\beta x, \alpha\beta w, \alpha\beta z, \beta y).$$

This map has 2 vertices $v_1 = \{(x, y, z, w), (\alpha x, \alpha w, \alpha z, \alpha y)\}$, $v_2 = \{(\alpha\beta x, \alpha\beta y, \beta z, \beta w), (\beta x, \alpha\beta w, \alpha\beta z, \beta y)\}$, 4 edges $e_1 = \{x, \alpha x, \beta x, \alpha\beta x\}$, $e_2 = \{y, \alpha y, \beta y, \alpha\beta y\}$, $e_3 = \{z, \alpha z, \beta z, \alpha\beta z\}$,

$e_4 = \{w, \alpha w, \beta w, \alpha\beta w\}$ and 2 faces $f_2 = \{(x, \alpha\beta y, z, \beta y, \alpha x, \alpha\beta w), (\beta x, \alpha w, \alpha\beta x, y, \beta z, \alpha y)\}$, $f_2 = \{(\beta w, \alpha z), (w, \alpha\beta z)\}$. Its Euler characteristic is

$$\chi(M) = 2 - 4 + 2 = 0$$

and $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. Thereby it is a map of $D_{0.4.0}$ on a Klein bottle with 2 faces accordant with its geometry.

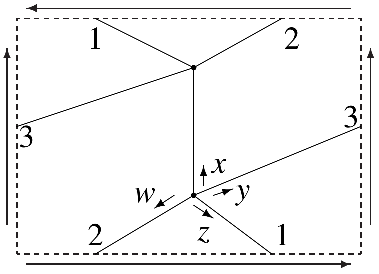


Fig.5.3.3

The following result was gotten by Tutte in [Tut2], which establishes a relation for embedded graphs with that of combinatorial maps.

Theorem 5.3.10 *For an embedded graph G on a locally orientable surface S, there exists one combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with an underlying graph G and for a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, there is an embedded graph G underlying M on S.*

Similar to the definition of a multi-voltage graph, we can define a multi-voltage map and its lifting by applying a multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ with $\Gamma_i = \Gamma_j$ for any integers $i, j, 1 \leq i, j \leq n$.

Definition 5.3.1 *Let $(\widetilde{\Gamma}; O)$ be a finite multi-group with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$, where $\Gamma = \{g_1, g_2, \cdots, g_m\}$ and an operation set $O(\widetilde{\Gamma}) = \{\circ_i | 1 \leq i \leq n\}$ and let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a combinatorial map. If there is a mapping $\psi : \mathcal{X}_{\alpha,\beta} \to \widetilde{\Gamma}$ such that*

(1) *for $\forall x \in \mathcal{X}_{\alpha,\beta}, \forall \sigma \in K = \{1, \alpha, \beta, \alpha\beta\}$, $\psi(\alpha x) = \psi(x)$, $\psi(\beta x) = \psi(\alpha\beta x) = \psi(x)^{-1}$;*

(2) *for any face $f = (x, y, \cdots, z)(\beta z, \cdots, \beta y, \beta x)$, $\psi(f, i) = \psi(x) \circ_i \psi(y) \circ_i \cdots \circ_i \psi(z)$, where $\circ_i \in O(\widetilde{\Gamma})$, $1 \leq i \leq n$ and $\langle \psi(f, i) | f \in F(v) \rangle = G$ for $\forall v \in V(G)$, where $F(v)$ denotes all faces incident with v,*

then the 2-tuple (M, ψ) is called a multi-voltage map.

The *lifting of a multi-voltage map* is defined by the next definition.

Definition 5.3.2 *For a multi-voltage map* (M,ψ), *the lifting map* $M^{\psi} = (\mathcal{X}^{\psi}_{\alpha^{\psi},\beta^{\psi}}, \mathcal{P}^{\psi})$ *is defined by*

$$\mathcal{X}^{\psi}_{\alpha^{\psi},\beta^{\psi}} = \{x_g | x \in \mathcal{X}_{\alpha,\beta}, g \in \widetilde{\Gamma}\},$$

$$\mathcal{P}^{\psi} = \prod_{g\in\widetilde{\Gamma}} \prod_{(x,y,\cdots,z)(\alpha z,\cdots,\alpha y,\alpha x)\in V(M)} (x_g, y_g, \cdots, z_g)(\alpha z_g, \cdots, \alpha y_g, \alpha x_g),$$

where

$$\alpha^{\psi} = \prod_{x\in\mathcal{X}_{\alpha,\beta}, g\in\widetilde{\Gamma}} (x_g, \alpha x_g), \quad \beta^{\psi} = \prod_{i=1}^{m} \prod_{x\in\mathcal{X}_{\alpha,\beta}} (x_{g_i}, (\beta x)_{g_i \circ_i \psi(x)})$$

with a convention that $(\beta x)_{g_i\circ_i\psi(x)} = y_{g_i}$ *for some quadricells* $y \in \mathcal{X}_{\alpha,\beta}$.

Notice that the lifting M^{ψ} is connected and $\Psi^{\psi}_I = \langle \alpha^{\psi}\beta^{\psi}, \mathcal{P}^{\psi}\rangle$ is transitive on $\mathcal{X}^{\psi}_{\alpha^{\psi},\beta^{\psi}}$ if and only if $\Psi_I = \langle \alpha\beta, \mathcal{P}\rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. We get a result in the following.

Theorem 5.3.11 *The Euler characteristic* $\chi(M^{\psi})$ *of the lifting map* M^{ψ} *of a multi-voltage map* $(M,\widetilde{\Gamma})$ *is*

$$\chi(M^{\psi}) = |\Gamma|\left(\chi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\left(\frac{1}{o(\psi(f,\circ_i))} - \frac{1}{n}\right)\right),$$

where $F(M)$ *and* $o(\psi(f,\circ_i))$ *denote the set of faces in* M *and the order of* $\psi(f,\circ_i)$ *in* $(\Gamma;\circ_i)$, *respectively.*

Proof By definition the lifting map M^{ϑ} has $|\Gamma|\nu(M)$ vertices, $|\Gamma|\varepsilon(M)$ edges. Notice that each lifting of the boundary walk of a face is a homogenous lifting by definition of β^{ψ}. Similar to the proof of Theorem 2.2.3, we know that M^{ϑ} has $\sum\limits_{i=1}^{n}\sum\limits_{f\in F(M)} \frac{|\Gamma|}{o(\psi(f,\circ_i))}$ faces. By the Eular-Poincaré formula we get that

$$\begin{aligned}\chi(M^{\psi}) &= \nu(M^{\psi}) - \varepsilon(M^{\psi}) + \phi(M^{\psi}) \\ &= |\Gamma|\nu(M) - |\Gamma|\varepsilon(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\frac{|\Gamma|}{o(\psi(f,\circ_i))} \\ &= |\Gamma|\left(\chi(M) - \phi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\frac{1}{o(\psi(f,\circ_i))}\right) \\ &= |G|\left(\chi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\left(\frac{1}{o(\psi(f,\circ_i))} - \frac{1}{n}\right)\right).\end{aligned}$$

□

Recently, more and more papers concentrated on finding *regular maps* on surface, which are related with *discrete groups, discrete geometry* and *crystal physics*. For this objective, an important way is by the voltage assignment technique on maps. See references [Mal1], [MNS1] and [NeS1]-[NeS1] for details. It is also an interesting problem to apply multi-voltage maps for finding non-regular or other maps with some constraint conditions.

Motivated by the Four Color Conjecture, Tait conjectured that *every simple 3-polytope is hamiltonian* in 1880. By Steinitz's a famous result (See [Gru1] for details), this conjecture is equivalent to that *every 3-connected cubic planar graph is hamiltonian*. Tutte disproved this conjecture by giving a 3-connected non-hamiltonian cubic planar graph with 46 vertices in 1946 and proved that *every* 4*-connected planar graph is hamiltonian* [Tut1] in 1956. In [Gru3], Grünbaum conjectured that *each* 4*-connected graph embeddable in the torus or in the projective plane is hamiltonian*. This conjecture had been solved for the projective plane case by Thomas and Yu [ThY1] in 1994. Notice that the splitting operator ϑ constructed in the proof of Theorem 2.2.10 is a planar operator. Applying Theorem 2.2.10 on surfaces we know that *for every map M on a surface, M^{ϑ} is non-hamiltonian*. In fact, we can further get an interesting result related with Tait's conjecture.

Theorem 5.3.12 *There exist infinite* $3-$*connected non-hamiltonian cubic maps on each locally orientable surface.*

Proof Notice that there exist 3-connected triangulations on every locally orientable surface S. Each dual of them is a 3-connected cubic map on S. Now we define a splitting operator σ as shown in Fig.5.3.4.

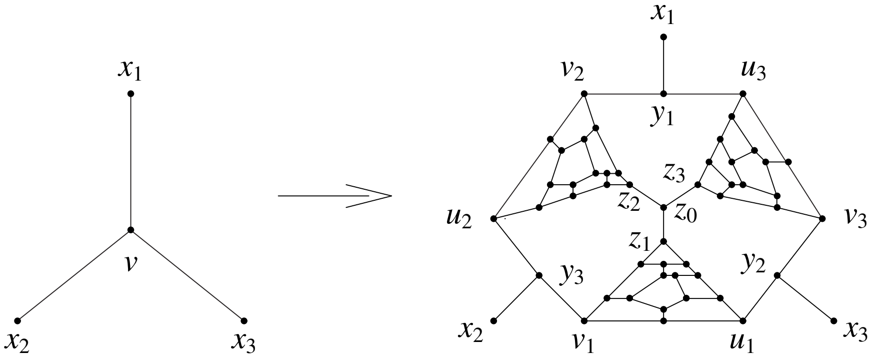


Fig.5.3.4

For a 3-connected cubic map M, we prove that $M^{\sigma(v)}$ is non-hamiltonian for $\forall v \in$

$V(M)$. According to Theorem 2.1.7, we only need to prove that there are no $y_1 - y_2$, or $y_1 - y_3$, or $y_2 - y_3$ hamiltonian path in the nucleus $N(\sigma(v))$ of operator σ.

Let $H(z_i)$ be a component of $N(\sigma(v))\backslash\{z_0z_i, y_{i-1}u_{i+1}, y_{i+1}v_{i-1}\}$ which contains the vertex $z_i, 1 \leq i \leq 3$(all these indices mod 3). If there exists a $y_1 - y_2$ hamiltonian path P in $N(\sigma(v))$, we prove that there must be a $u_i - v_i$ hamiltonian path in the subgraph $H(z_i)$ for an integer $i, 1 \leq i \leq 3$.

Since P is a hamiltonian path in $N(\sigma(v))$, there must be that $v_1y_3u_2$ or $u_2y_3v_1$ is a subpath of P. Now let $E_1 = \{y_1u_3, z_0z_3, y_2v_3\}$, we know that $|E(P)\bigcap E_1| = 2$. Since P is a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$, we must have $y_1u_3 \notin E(P)$ or $y_2v_3 \notin E(P)$. Otherwise, by $|E(P)\bigcap S_1| = 2$ we get that $z_0z_3 \notin E(P)$. But in this case, P can not be a $y_1 - y_2$ hamiltonian path in $N(\sigma(v))$, a contradiction.

Assume $y_2v_3 \notin E(P)$. Then $y_2u_1 \in E(P)$. Let $E_2 = \{u_1y_2, z_1z_0, v_1y_3\}$. We also know that $|E(P)\bigcap E_2| = 2$ by the assumption that P is a hamiltonian path in $N(\sigma(v))$. Hence $z_0z_1 \notin E(P)$ and the $v_1 - u_1$ subpath in P is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$.

Similarly, if $y_1u_3 \notin E(P)$, then $y_1v_2 \in E(P)$. Let $E_3 = \{y_1v_2, z_0z_2, y_3u_2\}$. We can also get that $|E(P)\bigcap E_3| = 2$ and a $v_2 - u_2$ hamiltonian path in the subgraph $H(z_2)$.

Now if there is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$, then the graph $H(z_1) + u_1v_1$ must be hamiltonian. According to the Grinberg's criterion for planar hamiltonian graphs, we know that

$$\phi'_3 - \phi"_3 + 2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) + 6(\phi'_8 - \phi"_8) = 0, \qquad (5-1)$$

where ϕ'_i or $\phi"_i$ is the number of i-gons in the interior or exterior of a chosen hamiltonian circuit C passing through u_1v_1 in the graph $H(z_1) + u_1v_1$. Since it is obvious that

$$\phi'_3 = \phi"_8 = 1, \quad \phi"_3 = \phi'_8 = 0,$$

we get that

$$2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) = 5, \qquad (5-2)$$

by (5-1). Because $\phi'_4 + \phi"_4 = 2$, so $\phi'_4 - \phi"_4 = 0, 2$ or -2. Now the valency of z_1 in $H(z_1)$ is 2, so the 4-gon containing the vertex z_1 must be in the interior of C, that is $\phi'_4 - \phi"_4 \neq -2$. If $\phi'_4 - \phi"_4 = 0$ or $\phi'_4 - \phi"_4 = 2$, we get $3(\phi'_5 - \phi"_5) = 5$ or $3(\phi'_5 - \phi"_5) = 1$, a contradiction.

Notice that $H(z_1) \simeq H(z_2) \simeq H(z_3)$. If there exists a $v_2 - u_2$ hamiltonian path in $H(z_2)$, a contradiction can be also gotten. So there does not exist a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$. Similarly , there are no $y_1 - y_3$ or $y_2 - y_3$ hamiltonian paths in the graph $N(\sigma(v))$. Whence, $M^{\sigma(v)}$ is non-hamiltonian.

Now let n be an integer, $n \geq 1$. We get that

$$\begin{aligned}
M_1 &= (M)^{\sigma(u)}, \quad u \in V(M);\\
M_2 &= (M_1)^{N(\sigma(v))(v)}, \quad v \in V(M_1);\\
\cdots & \cdots \cdots\cdots\cdots\cdots\cdots\cdots;\\
M_n &= (M_{n-1})^{N(\sigma(v))(w)}, \quad w \in V(M_{n-1});\\
\cdots & \cdots \cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}$$

All of these maps are 3-connected non-hamiltonian cubic maps on the surface S. This completes the proof. $\square$

Corollary 5.3.3 *There is not a locally orientable surface on which every 3-connected cubic map is hamiltonian.*

§5.4 MULTI-EMBEDDINGS OF GRAPHS

5.4.1 Multi-Surface Genus Range. Let $S_1, S_2, \cdots, S_k$ be k locally orientable surfaces and G a connected graph. Define numbers

$$\gamma(G; S_1, S_2, \cdots, S_k) = \min\left\{\sum_{i=1}^{k} \gamma(G_i) \middle| G = \biguplus_{i=1}^{k} G_i, G_i \rightarrow S_i, 1 \leq i \leq k\right\},$$

$$\gamma_M(G; S_1, S_2, \cdots, S_k) = \max\left\{\sum_{i=1}^{k} \gamma(G_i) \middle| G = \biguplus_{i=1}^{k} G_i, G_i \rightarrow S_i, 1 \leq i \leq k\right\}$$

and the *multi-genus range* $GR(G; S_1, S_2, \cdots, S_k)$ by

$$GR(G; S_1, S_2, \cdots, S_k) = \left\{\sum_{i=1}^{k} g(G_i) \middle| G = \biguplus_{i=1}^{k} G_i, G_i \rightarrow S_i, 1 \leq i \leq k\right\},$$

where G_i is embeddable on a surface of genus $g(G_i)$. Then we get the following result.

Theorem 5.4.1 *Let G be a connected graph and let $S_1, S_2, \cdots, S_k$ be locally orientable surfaces with empty overlapping. Then*

$$GR(G; S_1, S_2, \cdots, S_k) = [\gamma(G; S_1, S_2, \cdots, S_k), \gamma_M(G; S_1, S_2, \cdots, S_k)].$$

Proof Let $G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \leq i \leq k$. We prove that there are no gap in the multi-genus range from $\gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_k)$ to $\gamma_M(G_1) + \gamma_M(G_2) + \cdots + \gamma_M(G_k)$. According to Theorems 2.3.8 and 2.3.12, we know that the genus range $GR^O(G_i)$ or $GR^N(G)$ is $[\gamma(G_i), \gamma_M(G_i)]$ or $[\widetilde{\gamma}(G_i), \widetilde{\gamma}_M(G_i)]$ for any integer $i, 1 \leq i \leq k$. Whence, there exists a multi-embedding of G on k locally orientable surfaces $P_1, P_2, \cdots, P_k$ with $g(P_1) = \gamma(G_1)$, $g(P_2) = \gamma(G_2)$,$\cdots$, $g(P_k) = \gamma(G_k)$. Consider the graph G_1, then G_2, and then G_3, $\cdots$ to get multi-embedding of G on k locally orientable surfaces step by step. We get a multi-embedding of G on k surfaces with genus sum at least being an unbroken interval $[\gamma(G_1) + \gamma(G_2) + \cdots + \gamma(G_k), \gamma_M(G_1) + \gamma_M(G_2) + \cdots + \gamma_M(G_k)]$ of integers.

By definitions of $\gamma(G; S_1, S_2, \cdots, S_k)$ and $\gamma_M(G; S_1, S_2, \cdots, S_k)$, we assume that $G = \biguplus_{i=1}^{k} G_i', G_i' \to S_i, 1 \leq i \leq k$ and $G = \biguplus_{i=1}^{k} G_i'', G_i'' \to S_i, 1 \leq i \leq k$ attain the extremal values $\gamma(G; S_1, S_2, \cdots, S_k)$ and $\gamma_M(G; S_1, S_2, \cdots, S_k)$, respectively. Then we know that the multi-embedding of G on k surfaces with genus sum is at least an unbroken intervals $\left[\sum_{i=1}^{k} \gamma(G_i'), \sum_{i=1}^{k} \gamma_M(G_i')\right]$ and $\left[\sum_{i=1}^{k} \gamma(G_i''), \sum_{i=1}^{k} \gamma_M(G_i'')\right]$ of integers.

Since

$$\sum_{i=1}^{k} g(S_i) \in \left[\sum_{i=1}^{k} \gamma(G_i'), \sum_{i=1}^{k} \gamma_M(G_i')\right] \bigcap \left[\sum_{i=1}^{k} \gamma(G_i''), \sum_{i=1}^{k} \gamma_M(G_i'')\right],$$

we get that

$$GR(G; S_1, S_2, \cdots, S_k) = [\gamma(G; S_1, S_2, \cdots, S_k), \gamma_M(G; S_1, S_2, \cdots, S_k)].$$

This completes the proof. □

Furthermore, we get the following result for multi-surface embeddings of complete graphs.

Theorem 5.4.2 *Let $P_1, P_2, \cdots, P_k$ and $Q_1, Q_2, \cdots, Q_k$ be respective k orientable and non-orientable surfaces of genus$\geq$ 1. A complete graph K_n is multi-surface embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping if and only if*

$$\sum_{i=1}^{k} \left\lceil \frac{3 + \sqrt{16g(P_i) + 1}}{2} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor \frac{7 + \sqrt{48g(P_i) + 1}}{2} \right\rfloor$$

and is multi-surface embeddable in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping if and only if

$$\sum_{i=1}^{k} \left\lceil 1 + \sqrt{2g(Q_i)} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor \frac{7 + \sqrt{24g(Q_i) + 1}}{2} \right\rfloor.$$

Proof According to Theorems 5.3.4-5.3.9, we know that the genus $g(P)$ of an orientable surface P on which a complete graph K_n is embeddable satisfies

$$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \leq g(P) \leq \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor,$$

i.e.,

$$\frac{(n-3)(n-4)}{12} \leq g(P) \leq \frac{(n-1)(n-2)}{4}.$$

If $g(P) \geq 1$, we get that

$$\left\lceil \frac{3+\sqrt{16g(P)+1}}{2} \right\rceil \leq n \leq \left\lfloor \frac{7+\sqrt{48g(P)+1}}{2} \right\rfloor.$$

Similarly, if K_n is embeddable on a non-orientable surface Q, then

$$\left\lceil \frac{(n-3)(n-4)}{6} \right\rceil \leq g(Q) \leq \left\lfloor \frac{(n-1)^2}{2} \right\rfloor,$$

i.e.,

$$\left\lceil 1+\sqrt{2g(Q)} \right\rceil \leq n \leq \left\lfloor \frac{7+\sqrt{24g(Q)+1}}{2} \right\rfloor.$$

Now if K_n is multi-surface embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping, then there must exists a partition $n = n_1 + n_2 + \cdots + n_k$, $n_i \geq 1, 1 \leq i \leq k$. Since each vertex-induced subgraph of a complete graph is still a complete graph, we know that for any integer $i, 1 \leq i \leq k$,

$$\left\lceil \frac{3+\sqrt{16g(P_i)+1}}{2} \right\rceil \leq n_i \leq \left\lfloor \frac{7+\sqrt{48g(P_i)+1}}{2} \right\rfloor.$$

Whence, we know that

$$\sum_{i=1}^{k} \left\lceil \frac{3+\sqrt{16g(P_i)+1}}{2} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor \frac{7+\sqrt{48g(P_i)+1}}{2} \right\rfloor. \qquad (5-3)$$

On the other hand, if the inequality (5-3) holds, we can find positive integers $n_1, n_2, \cdots, n_k$ with $n = n_1 + n_2 + \cdots + n_k$ and

$$\left\lceil \frac{3+\sqrt{16g(P_i)+1}}{2} \right\rceil \leq n_i \leq \left\lfloor \frac{7+\sqrt{48g(P_i)+1}}{2} \right\rfloor.$$

for any integer $i, 1 \leq i \leq k$. This enables us to establish a partition $K_n = \biguplus_{i=1}^{k} K_{n_i}$ for K_n and embed each K_{n_i} on P_i for $1 \leq i \leq k$. Therefore, we get a multi-embedding of K_n in $P_1, P_2, \cdots, P_k$ with empty overlapping.

Similarly, if K_n is multi-surface embeddable in $Q_1, Q_2, \cdots Q_k$ with empty overlapping, there must exists a partition $n = m_1 + m_2 + \cdots + m_k$, $m_i \geq 1, 1 \leq i \leq k$ and

$$\left\lceil 1 + \sqrt{2g(Q_i)} \right\rceil \leq m_i \leq \left\lfloor \frac{7 + \sqrt{24g(Q_i) + 1}}{2} \right\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Whence, we get that

$$\sum_{i=1}^{k} \left\lceil 1 + \sqrt{2g(Q_i)} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor \frac{7 + \sqrt{24g(Q_i) + 1}}{2} \right\rfloor. \qquad (5-4)$$

Now if the inequality (5-4) holds, we can also find positive integers $m_1, m_2, \cdots, m_k$ with $n = m_1 + m_2 + \cdots + m_k$ and

$$\left\lceil 1 + \sqrt{2g(Q_i)} \right\rceil \leq m_i \leq \left\lfloor \frac{7 + \sqrt{24g(Q_i) + 1}}{2} \right\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Similar to those of orientable cases, we get a multi-surfaces embedding of K_n in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping. $\square$

Corollary 5.4.1 *A complete graph K_n is multi-surface embeddable in $k, k \geq 1$ orientable surfaces of genus $p, p \geq 1$ with empty overlapping if and only if*

$$\left\lceil \frac{3 + \sqrt{16p + 1}}{2} \right\rceil \leq \frac{n}{k} \leq \left\lfloor \frac{7 + \sqrt{48p + 1}}{2} \right\rfloor$$

and is multi-surface embeddable in $l, l \geq 1$ non-orientable surfaces of genus $q, q \geq 1$ with empty overlapping if and only if

$$\left\lceil 1 + \sqrt{2q} \right\rceil \leq \frac{n}{k} \leq \left\lfloor \frac{7 + \sqrt{24q + 1}}{2} \right\rfloor.$$

Corollary 5.4.2 *A complete graph K_n is multi-surface embeddable in $s, s \geq 1$ tori with empty overlapping if and only if*

$$4s \leq n \leq 7s$$

and is multi-surface embeddable in $t, t \geq 1$ projective planes with empty overlapping if and only if

$$3t \leq n \leq 6t.$$

Similarly, the following result holds for complete bipartite graphs $K(n, n)$, $n \geq 1$.

Theorem 5.4.3 *Let* $P_1, P_2, \cdots, P_k$ *and* $Q_1, Q_2, \cdots, Q_k$ *be respective* k *orientable and* k *non-orientable surfaces of genus*≥ 1. *A complete bipartite graph* $K(n, n)$ *is multi-surface embeddable in* $P_1, P_2, \cdots, P_k$ *with empty overlapping if and only if*

$$\sum_{i=1}^{k} \left\lceil 1 + \sqrt{2g(P_i)} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor 2 + 2\sqrt{g(P_i)} \right\rfloor$$

and is multi-surface embeddable in $Q_1, Q_2, \cdots, Q_k$ *with empty overlapping if and only if*

$$\sum_{i=1}^{k} \left\lceil 1 + \sqrt{g(Q_i)} \right\rceil \leq n \leq \sum_{i=1}^{k} \left\lfloor 2 + \sqrt{2g(Q_i)} \right\rfloor.$$

Proof Similar to the proof of Theorem 5.4.2, we get the result. □

5.4.2 Classification of Manifold Graph. By Theorem 5.2.1, we can give a combinatorial definition for a graph embedded in an n-manifold, i.e., a *manifold graph* similar to that the Tutte's definition for combinatorial maps.

Definition 5.4.1 *For any integer* $n, n \geq 2$, *an* n*-dimensional manifold graph* ${}^n\mathcal{G}$ *is a pair* ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *in where a permutation* $\mathcal{L}$ *acting on* $\mathcal{E}_\Gamma$ *of a disjoint union* $\Gamma x = \{\sigma x | \sigma \in \Gamma\}$ *for* $\forall x \in E$, *where* E *is a finite set and* $\Gamma = \{\mu, o | \mu^2 = o^n = 1, \mu o = o\mu\}$ *is a commutative group of order* $2n$ *with the following conditions hold:*

(1) $\forall x \in \mathcal{E}_K$, *there does not exist an integer* k *such that* $\mathcal{L}^k x = o^i x$ *for* $\forall i, 1 \leq i \leq n-1$*;*

(2) $\mu\mathcal{L} = \mathcal{L}^{-1}\mu$;

(3) *The group* $\Psi_J = \langle \mu, o, \mathcal{L} \rangle$ *is transitive on* $\mathcal{E}_\Gamma$.

According to conditions (1) and (2), a *vertex* v *of an* n*-dimensional manifold graph* is defined to be an n-tuple

$$\{(o^i x_1, o^i x_2, \cdots, o^i x_{s_l(v)})(o^i y_1, o^i y_2, \cdots, o^i y_{s_2(v)}) \cdots (o^i z_1, o^i z_2, \cdots, o^i z_{s_{l(v)}(v)}); 1 \leq i \leq n\}$$

of permutations of $\mathcal{L}$ action on $\mathcal{E}_\Gamma$, edges to be these orbits of Γ action on $\mathcal{E}_\Gamma$. The number $s_1(v) + s_2(v) + \cdots + s_{l(v)}(v)$ is called the *valency of* v, denoted by $\rho_G^{s_1, s_2, \cdots, s_{l(v)}}(v)$. The condition (*iii*) is used to ensure that an n-dimensional manifold graph is connected. Comparing definitions of a map with that of n-dimensional manifold graph, the following result holds.

Theorem 5.4.4 *For any integer $n, n \geq 2$, every n-dimensional manifold graph ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ is correspondent to a unique map $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ in which each vertex v in ${}^n\mathcal{G}$ is converted to $l(v)$ vertices $v_1, v_2, \cdots, v_{l(v)}$ of M. Conversely, a map $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ is also correspondent to an n-dimensional manifold graph ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in which $l(v)$ vertices $u_1, u_2, \cdots, u_{l(v)}$ of M are converted to one vertex u of ${}^n\mathcal{G}$.*

Two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are said to be *isomorphic* if there exists a one-to-one mapping $\kappa : \mathcal{E}^1_{\Gamma_1} \rightarrow \mathcal{E}^2_{\Gamma_2}$ such that $\kappa\mu = \mu\kappa, \kappa o = o\kappa$ and $\kappa\mathcal{L}_1 = \mathcal{L}_2\kappa$. If $\mathcal{E}^1_{\Gamma_1} = \mathcal{E}^2_{\Gamma_2} = \mathcal{E}_\Gamma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, an isomorphism between ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$ is called an automorphism of ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$. It is immediately that all automorphisms of ${}^n\mathcal{G}$ form a group under the composition operation. We denote this group by $\mathrm{Aut}^n\mathcal{G}$.

It is clear that for two isomorphic n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, their underlying graphs G_1 and G_2 are isomorphic. For an embedding ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in an n-dimensional manifold and $\forall\zeta \in \mathrm{Aut}_{\frac{1}{2}}G$, an induced action of ζ on $\mathcal{E}_\Gamma$ is defined by $\zeta(gx) = g\zeta(x)$ for $\forall x \in \mathcal{E}_\Gamma$ and $\forall g \in \Gamma$. Then the following result holds.

Theorem 5.4.5 $\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle$.

Proof First we prove that two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are isomorphic if and only if there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$ or $\mathcal{L}_2^{-1}$.

If there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$, then the n-dimensional manifold graph ${}^n\mathcal{G}_1$ is isomorphic to ${}^n\mathcal{G}_2$ by definition. If $\mathcal{L}_1^\zeta = \mathcal{L}_2^{-1}$, then $\mathcal{L}_1^{\zeta\mu} = \mathcal{L}_2$. The n-dimensional manifold graph ${}^n\mathcal{G}_1$ is also isomorphic to ${}^n\mathcal{G}_2$.

By the definition of isomorphism ξ between n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, we know that

$$\mu\xi(x) = \xi\mu(x),\ o\xi(x) = \xi o(x) \text{ and } \mathcal{L}_1^\xi(x) = \mathcal{L}_2(x)$$

for $\forall x \in \mathcal{E}_\Gamma$. By definition these conditions

$$o\xi(x) = \xi o(x) \text{ and } \mathcal{L}_1^\xi(x) = \mathcal{L}_2(x)$$

are just the condition of an automorphism ξ or $\alpha\xi$ on $X_{\frac{1}{2}}(\Gamma)$. Whence, the assertion is true.

Now let $\mathcal{E}^1_{\Gamma_1} = \mathcal{E}^2_{\Gamma_2} = \mathcal{E}_\Gamma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$. We know that

$$\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle\,. \qquad \square$$

Similarly, the action of an automorphism of manifold graph on $\mathcal{E}_\Gamma$ is fixed-free shown in the following.

Theorem 5.4.6 *Let* $^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *be an n-dimensional manifold graph. Then* $(\mathrm{Aut}^n\mathcal{G})_x$ *is trivial for* $\forall x \in \mathcal{E}_\Gamma$.

Proof For $\forall g \in (\mathrm{Aut}^n\mathcal{G})_x$, we prove that $g(y) = y$ for $\forall y \in \mathcal{E}_\Gamma$. In fact, since the group $\Psi_J = \langle \mu, o, \mathcal{L}\rangle$ is transitive on $\mathcal{E}_\Gamma$, there exists an element $\tau \in \Psi_J$ such that $y = \tau(x)$. By definition we know that every element in Ψ_J is commutative with automorphisms of $^n\mathcal{G}$. Whence, we get that

$$g(y) = g(\tau(x)) = \tau(g(x)) = \tau(x) = y,$$

i.e., $(\mathrm{Aut}^n\mathcal{G})_x$ is trivial. □

Corollary 5.4.3 *Let* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ *be a map. Then for* $\forall x \in \mathcal{X}_{\alpha,\beta}$, $(\mathrm{Aut}M)_x$ *is trivial.*

For an n-dimensional manifold graph $^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$, an $x \in \mathcal{E}_\Gamma$ is said a *root* of $^n\mathcal{G}$. If we have chosen a root r on an n-dimensional manifold graph $^n\mathcal{G}$, then $^n\mathcal{G}$ is called a *rooted n-dimensional manifold graph*, denoted by $^n\mathcal{G}^r$. Two rooted n-dimensional manifold graphs $^n\mathcal{G}^{r_1}$ and $^n\mathcal{G}^{r_2}$ are said to be *isomorphic* if there is an isomorphism ς between them such that $\varsigma(r_1) = r_2$. Applying Theorem 5.4.6 and Corollary 5.2.1, we get an enumeration result for n-dimensional manifold graphs underlying a graph G following.

Theorem 5.4.7 *For any integer* $n, n \geq 3$, *the number* $r_n^S(G)$ *of rooted n-dimensional manifold graphs underlying a graph G is*

$$r_n^S(G) = \frac{n\varepsilon(G) \prod\limits_{v\in V(G)} \rho_G(v)!}{|Aut_{\frac{1}{2}}G|}.$$

Proof Denote the set of all non-isomorphic n-dimensional manifold graphs underlying a graph G by $\mathcal{G}^S(G)$. For an n-dimensional graph $^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L}) \in \mathcal{G}^S(G)$, denote the number of non-isomorphic rooted n-dimensional manifold graphs underlying $^n\mathcal{G}$ by $r(^n\mathcal{G})$. By a result in permutation groups theory, for $\forall x \in \mathcal{E}_\Gamma$ we know that

$$|\mathrm{Aut}^n\mathcal{G}| = |(\mathrm{Aut}^n\mathcal{G})_x||x^{\mathrm{Aut}^n\mathcal{G}}|.$$

According to Theorem 2.3.23, $|(\mathrm{Aut}^n\mathcal{G})_x| = 1$. Whence, $|x^{\mathrm{Aut}^n\mathcal{G}}| = |\mathrm{Aut}^n\mathcal{G}|$. However there are $|\mathcal{E}_\Gamma| = 2n\varepsilon(G)$ roots in $^n\mathcal{G}$ by definition. Therefore, the number of non-isomorphic rooted n-dimensional manifold graphs underlying an n-dimensional graph ${}^n\mathcal{G}$ is

$$r({}^n\mathcal{G}) = \frac{|\mathcal{E}_\Gamma|}{|\mathrm{Aut}^n\mathcal{G}|} = \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Whence, the number of non-isomorphic rooted n-dimensional manifold graphs underlying a graph G is

$$r_n^S(G) = \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

According to Theorem 5.4.5, $\mathrm{Aut}^n\mathcal{G} \leq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle$. Whence $\tau \in \mathrm{Aut}^n\mathcal{G}$ for ${}^n\mathcal{G} \in \mathcal{G}^S(G)$ if and only if $\tau \in (\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}$. Therefore, we know that $\mathrm{Aut}^n\mathcal{G} = (\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}$. Because of $|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle| = |(\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle)_{{}^n\mathcal{G}}||{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}|$, we get that

$$|{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}| = \frac{2|\mathrm{Aut}_{\frac{1}{2}}G|}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Therefore,

$$\begin{aligned} r_n^S(G) &= \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle|} \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle|}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle|} \sum_{{}^n\mathcal{G}\in\mathcal{G}^S(G)} |{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}| \\ &= \frac{n\varepsilon(G)\prod\limits_{v\in V(G)}\rho_G(v)!}{|\mathrm{Aut}_{\frac{1}{2}}G|} \end{aligned}$$

by applying Corollary 5.2.1. □

Notice the fact that an embedded graph in 2-dimensional manifold is just a map and Definition 5.4.1 turn to Tutte's definition for combinatorial map. We can also get an enumeration result for rooted maps on surfaces underlying a graph G by applying Theorems 5.3.2 and 5.4.6 following.

Theorem 5.4.8([MaL4]) *The number $r^L(\Gamma)$ of rooted maps on locally orientable surfaces underlying a connected graph G is*

$$r^L(G) = \frac{2^{\beta(G)+1}\varepsilon(G)\prod\limits_{v\in V(G)}(\rho(v)-1)!}{|Aut_{\frac{1}{2}}G|},$$

where $\beta(G) = \varepsilon(G) - \nu(G) + 1$ *is the Betti number of* G.

Similarly, for a graph $G = \bigoplus\limits_{i=1}^{l} G_i$ and a multi-manifold $\widetilde{M} = \bigcup\limits_{i=1}^{l} \mathbf{M}^{l_i}$, choose l commutative groups $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$, where $\Gamma_i = \left\langle \mu_i, o_i | \mu_i^2 = o^{h_i} = 1 \right\rangle$ for any integer $i, 1 \le i \le l$. Consider permutations acting on $\bigcup\limits_{i=1}^{l} \mathcal{E}_{\Gamma_i}$, where for any integer $i, 1 \le i \le l$, $\mathcal{E}_{\Gamma_i}$ is a disjoint union $\Gamma_i x = \{\sigma_i x | \sigma_i \in \Gamma\}$ for $\forall x \in E(G_i)$. Similar to Definition 5.4.1, we can also get a multi-embedding of G in $\widetilde{M} = \bigcup\limits_{i=1}^{l} \mathbf{M}^{h_i}$.

§5.5 GRAPH PHASE SPACES

5.5.1 Graph Phase. For convenience, we first introduce some notations used in this section in the following.

$\widetilde{\mathbf{M}}$ – A multi-manifold $\widetilde{\mathbf{M}} = \bigcup\limits_{i=1}^{n} \mathbf{M}^{n_i}$, where each $\mathbf{M}^{n_i}$ is an n_i-manifold, $n_i \ge 2$.

$\overline{u} \in \widetilde{\mathbf{M}}$ – A point $\overline{u}$ of $\widetilde{\mathbf{M}}$.

$\mathcal{G}$ – A graph G embedded in $\widetilde{\mathbf{M}}$.

$C(\widetilde{\mathbf{M}})$ – The set of differentiable mappings $\omega : \widetilde{\mathbf{M}} \to \widetilde{\mathbf{M}}$ at each point $\overline{u}$ in $\widetilde{\mathbf{M}}$.

Now we define the phase of graph in a multi-space following.

Definition 5.5.1 *Let* $\mathcal{G}$ *be a graph embedded in a multi-manifold* $\widetilde{\mathbf{M}}$. *A phase of* $\mathcal{G}$ *in* $\widetilde{\mathbf{M}}$ *is a triple* $(\mathcal{G}; \omega, \Lambda)$ *with an operation* $\circ$ *on* $C(\widetilde{M})$, *where* $\omega : V(G) \to C(\widetilde{\mathbf{M}})$ *and* $\Lambda : E(\mathcal{G}) \to C(\widetilde{\mathbf{M}})$ *such that* $\Lambda(\overline{u}, \overline{v}) = \dfrac{\omega(\overline{u}) \circ \omega(\overline{v})}{\| \overline{u} - \overline{v} \|}$ *for* $\forall (\overline{u}, \overline{v}) \in E(\mathcal{G})$, *where* $\| \overline{u} \|$ *denotes the norm of* $\overline{u}$.

For examples, the complete graph K_4 embedded in $\mathbf{R}^3$ has a phase as shown in Fig.5.5.1, where $g \in C(\mathbf{R}^3)$ and $h \in C(\mathbf{R}^3)$.

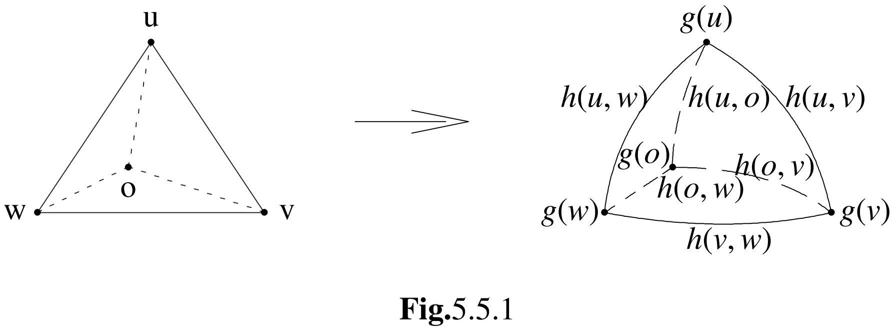


Fig.5.5.1

Similar to the adjacent matrix of graph, we can also define matrixes on graph phases.

Definition 5.5.2 *Let $(\mathcal{G};\omega,\Lambda)$ be a phase and $A[G]=[a_{ij}]_{p\times p}$ the adjacent matrix of a graph G with $V(G)=\{v_1,v_2,\cdots,v_p\}$. Define matrixes $V[\mathcal{G}]=[V_{ij}]_{p\times p}$ and $\Lambda[\mathcal{G}]=[\Lambda_{ij}]_{p\times p}$ by*

$$V_{ij}=\frac{\omega(\overline{v}_i)}{\parallel \overline{v}_i-\overline{v}_j \parallel}\ if\ a_{ij}\neq 0;\ otherwise, V_{ij}=0$$

and

$$\Lambda_{ij}=\frac{\omega(\overline{v}_i)\circ\omega(\overline{v}_j)}{\parallel \overline{v}_i-\overline{v}_j \parallel^2}\ if\ a_{ij}\neq 0;\ otherwise, \Lambda_{ij}=0,$$

where "$\circ$" is an operation on $C(\widetilde{M})$.

For example, for the phase of K_4 in Fig.5.5.1, if choice $g(u)=(x_1,x_2,x_3)$, $g(v)=(y_1,y_2,y_3)$, $g(w)=(z_1,z_2,z_3)$, $g(o)=(t_1,t_2,t_3)$ and $\circ=\times$, the multiplication of vectors in $\mathbf{R}^3$, then we get that

$$V(\mathcal{G})=\begin{bmatrix} 0 & \frac{g(u)}{\rho(u,v)} & \frac{g(u)}{\rho(u,w)} & \frac{g(u)}{\rho(u,o)} \\ \frac{g(v)}{\rho(v,u)} & 0 & \frac{g(v)}{\rho(v,w)} & \frac{g(v)}{\rho(v,t)} \\ \frac{g(w)}{\rho(w,u)} & \frac{g(w)}{\rho(w,v)} & 0 & \frac{g(w)}{\rho(w,o)} \\ \frac{g(o)}{\rho(o,u)} & \frac{g(o)}{\rho(o,v)} & \frac{g(o)}{\rho(o,w)} & 0 \end{bmatrix},$$

where,

$$\rho(u,v)=\rho(v,u)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2},$$

$$\rho(u,w)=\rho(w,u)=\sqrt{(x_1-z_1)^2+(x_2-z_2)^2+(x_3-z_3)^2},$$

$$\rho(u,o)=\rho(o,u)=\sqrt{(x_1-t_1)^2+(x_2-t_2)^2+(x_3-t_3)^2},$$

$$\rho(v,w)=\rho(w,v)=\sqrt{(y_1-z_1)^2+(y_2-z_2)^2+(y_3-z_3)^2},$$

$$\rho(v,o)=\rho(o,v)=\sqrt{(y_1-t_1)^2+(y_2-t_2)^2+(y_3-t_3)^2},$$

$$\rho(w,o)=\rho(o,w)=\sqrt{(z_1-t_1)^2+(z_2-t_2)^2+(z_3-t_3)^2}$$

and

$$\Lambda(\mathcal{G})=\begin{bmatrix} 0 & \frac{g(u)\times g(v)}{\rho^2(u,v)} & \frac{g(u)\times g(w)}{\rho^2(u,w)} & \frac{g(u)\times g(o)}{\rho^2(u,o)} \\ \frac{g(v)\times g(u)}{\rho^2(v,u)} & 0 & \frac{g(v)\times g(w)}{\rho^2(v,w)} & \frac{g(v\times g(o)}{\rho^2(v,o)} \\ \frac{g(w)\times g(u)}{\rho^2(w,u)} & \frac{g(w)\times g(v)}{\rho^2(w,v)} & 0 & \frac{g(w)\times g(o)}{\rho^2(w,o)} \\ \frac{g(o)\times g(u)}{\rho^2(o,u)} & \frac{g(o)\times g(v)}{\rho^2(o,v)} & \frac{g(o)\times g(w)}{\rho^2(o,w)} & 0 \end{bmatrix},$$

where,

$$\begin{aligned}
g(u)\times g(v) &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),\\
g(u)\times g(w) &= (x_2z_3 - x_3z_2, x_3z_1 - x_1z_3, x_1z_2 - x_2z_1),\\
g(u)\times g(o) &= (x_2t_3 - x_3t_2, x_3t_1 - x_1t_3, x_1t_2 - x_2t_1),\\
g(v)\times g(u) &= (y_2x_3 - y_3x_2, y_3x_1 - y_1x_3, y_1x_2 - y_2x_1),\\
g(v)\times g(w) &= (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1),\\
g(v)\times g(o) &= (y_2t_3 - y_3t_2, y_3t_1 - y_1t_3, y_1t_2 - y_2t_1),\\
g(w)\times g(u) &= (z_2x_3 - z_3x_2, z_3x_1 - z_1x_3, z_1x_2 - z_2x_1),\\
g(w)\times g(v) &= (z_2y_3 - z_3y_2, z_3y_1 - z_1y_3, z_1y_2 - z_2y_1),\\
g(w)\times g(o) &= (z_2t_3 - z_3t_2, z_3t_1 - z_1t_3, z_1t_2 - z_2t_1),\\
g(o)\times g(u) &= (t_2x_3 - t_3x_2, t_3x_1 - t_1x_3, t_1x_2 - t_2x_1),\\
g(o)\times g(v) &= (t_2y_3 - t_3y_2, t_3y_1 - t_1y_3, t_1y_2 - t_2y_1),\\
g(o)\times g(w) &= (t_2z_3 - t_3z_2, t_3z_1 - t_1z_3, t_1z_2 - t_2z_1).
\end{aligned}$$

For two given matrixes $A = [a_{ij}]_{p\times p}$ and $B = [b_{ij}]_{p\times p}$, the *star product* "$*$" on an operation "$\circ$" is defined by $A * B = [a_{ij} \circ b_{ij}]_{p\times p}$. We get the following result for matrixes $V[\mathcal{G}]$ and $\Lambda[\mathcal{G}]$.

Theorem 5.5.1 $V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]$.

Proof Calculation shows that each (i, j) entry in $V[\mathcal{G}] * V^t[\mathcal{G}]$ is

$$\frac{\omega(\overline{v}_i)}{\parallel \overline{v}_i - \overline{v}_j \parallel} \circ \frac{\omega(\overline{v}_j)}{\parallel \overline{v}_j - \overline{v}_i \parallel} = \frac{\omega(\overline{v}_i)\circ\omega(\overline{v}_j)}{\parallel \overline{v}_i - \overline{v}_j \parallel^2} = \Lambda_{ij},$$

where $1 \leq i, j \leq p$. Therefore, we get that

$$V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]. \qquad \square$$

An operation on graph phases called *addition* is defined in the following.

Definition 5.5.3 *For two phase spaces $(\mathcal{G}_1; \omega_1, \Lambda_1)$, $(\mathcal{G}_2; \omega_2, \Lambda_2)$ of graphs G_1, G_2 in $\widetilde{M}$ and two operations "$\bullet$" and "$\circ$" on $C(\widetilde{M})$, their addition is defined by*

$$(\mathcal{G}_1; \omega_1, \Lambda_1)\bigoplus(\mathcal{G}_2; \omega_2, \Lambda_2) = (\mathcal{G}_1 \bigoplus \mathcal{G}_2; \omega_1 \bullet \omega_2, \Lambda_1 \bullet \Lambda_2),$$

where $\omega_1 \bullet \omega_2 : V(\mathcal{G}_1 \bigcup \mathcal{G}_2) \to C(\widetilde{M})$ satisfying

$$\omega_1 \bullet \omega_2(\overline{u}) = \begin{cases} \omega_1(\overline{u}) \bullet \omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1) \bigcap V(\mathcal{G}_2), \\ \omega_1(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1) \setminus V(\mathcal{G}_2), \\ \omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_2) \setminus V(\mathcal{G}_1). \end{cases}$$

and

$$\Lambda_1 \bullet \Lambda_2(\overline{u}, \overline{v}) = \frac{\omega_1 \bullet \omega_2(\overline{u}) \circ \omega_1 \bullet \omega_2(\overline{v})}{\| \overline{u} - \overline{v} \|^2}$$

for $(\overline{u}, \overline{v}) \in E(\mathcal{G}_1) \bigcup E(\mathcal{G}_2)$.

The following result is immediately gotten by Definition 5.5.3.

Theorem 5.5.2 *For two given operations "•" and "∘" on* $C(\widetilde{M})$*, all graph phases in* $\widetilde{M}$ *form a linear space on the field* Z_2 *with a phase* $\bigoplus$ *for any graph phases* $(\mathcal{G}_1; \omega_1, \Lambda_1)$ *and* $(\mathcal{G}_2; \omega_2, \Lambda_2)$ *in* $\widetilde{M}$.

5.5.2 Graph Phase Transformation. The transformation of graph phase is defined in the following.

Definition 5.5.4 *Let* $(\mathcal{G}_1; \omega_1, \Lambda_1)$ *and* $(\mathcal{G}_2; \omega_2, \Lambda_2)$ *be graph phases of graphs* G_1 *and* G_2 *in a multi-space* $\widetilde{M}$ *with operations "*$\circ_1, \circ_2$*", respectively. If there exists a smooth mapping* $\tau \in C(\widetilde{M})$ *such that*

$$\tau : (\mathcal{G}_1; \omega_1, \Lambda_1) \to (\mathcal{G}_2; \omega_2, \Lambda_2),$$

i.e., for $\forall \overline{u} \in V(\mathcal{G}_1)$, $\forall (\overline{u}, \overline{v}) \in E(\mathcal{G}_1)$, $\tau(\mathcal{G}_1) = \mathcal{G}_2$, $\tau(\omega_1(\overline{u})) = \omega_2(\tau(\overline{u}))$ *and* $\tau(\Lambda_1(\overline{u}, \overline{v})) = \Lambda_2(\tau(\overline{u}, \overline{v}))$*, then we say* $(\mathcal{G}_1; \omega_1, \Lambda_1)$ *and* $(\mathcal{G}_2; \omega_2, \Lambda_2)$ *are transformable and* τ *a transform mapping.*

For examples, a projection p transforming an embedding of K_4 in $\mathbf{R}^3$ on the plane $\mathbf{R}^2$ is shown in Fig.5.5.2

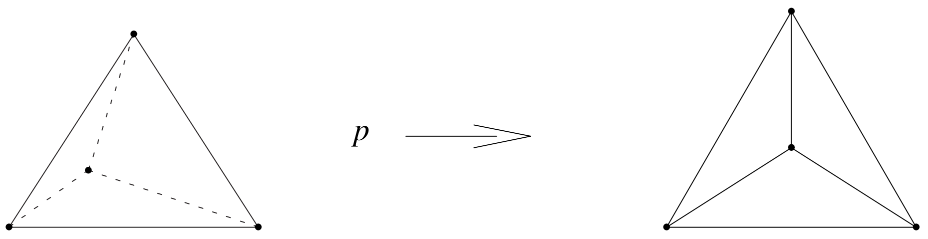


Fig.5.5.2

Theorem 5.5.3 *Let* $(\mathcal{G}_1;\omega_1,\Lambda_1)$ *and* $(\mathcal{G}_2;\omega_2,\Lambda_2)$ *be transformable graph phases with transform mapping* τ. *If* τ *is one-to-one on* $\mathcal{G}_1$ *and* $\mathcal{G}_2$, *then* $\mathcal{G}_1$ *is isomorphic to* $\mathcal{G}_2$.

Proof By definitions, if τ is one-to-one on $\mathcal{G}_1$ and $\mathcal{G}_2$, then τ is an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. □

A useful case in transformable graph phases is that one can find parameters $t_1, t_2, \cdots, t_q$, $q \geq 1$ such that each vertex of a graph phase is a smooth mapping of $t_1, t_2, \cdots, t_q$, i.e., for $\forall \overline{u} \in \widetilde{M}$, we consider it as $\overline{u}(t_1, t_2, \cdots, t_q)$. In this case, we introduce two conceptions on graph phases.

Definition 5.5.5 *For a graph phase* $(\mathcal{G};\omega,\Lambda)$, *define its capacity* $Ca(\mathcal{G};\omega,\Lambda)$ *and entropy* $En(\mathcal{G};\omega,\Lambda)$ *by*

$$Ca(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u})$$

and

$$En(\mathcal{G};\omega,\Lambda) = \log\left(\prod_{\overline{u}\in V(\mathcal{G})} \parallel \omega(\overline{u}) \parallel\right).$$

Then we know the following result.

Theorem 5.5.4 *For a graph phase* $(\mathcal{G};\omega,\Lambda)$, *its capacity* $Ca(\mathcal{G};\omega,\Lambda)$ *and entropy* $En(\mathcal{G};\omega,\Lambda)$ *satisfy the following differential equations*

$$\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) = \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i \ \ \text{and} \ \ \mathrm{d}En(\mathcal{G};\omega,\Lambda) = \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i,$$

where we use the Einstein summation convention, i.e., a sum is over i if it is appearing both in upper and lower indices.

Proof Not loss of generality, we assume $\overline{u} = (u_1, u_2, \cdots, u_p)$ for $\forall \overline{u} \in \widetilde{M}$. According to the invariance of differential form, we know that

$$\mathrm{d}\omega = \frac{\partial \omega}{\partial u_i}\mathrm{d}u_i.$$

By the definition of the capacity $Ca(\mathcal{G};\omega,\Lambda)$ and entropy $En(\mathcal{G};\omega,\Lambda)$ of a graph phase, we get that

$$\begin{aligned}
\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) &= \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\omega(\overline{u})) = \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial \omega(\overline{u})}{\partial u_i}\mathrm{d}u_i \\
&= \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u}))}{\partial u_i}\mathrm{d}u_i = \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i.
\end{aligned}$$

Similarly, we also obtain that

$$\begin{aligned} \mathrm{d}En(\mathcal{G};\omega,\Lambda) &= \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\log \parallel \omega(\overline{u}) \parallel) = \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial \log|\omega(\overline{u})|}{\partial u_i}\mathrm{d}u_i \\ &= \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \log \parallel \omega(\overline{u}) \parallel)}{\partial u_i}\mathrm{d}u_i = \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i. \end{aligned}$$

This completes the proof. □

For the 3-dimensional Euclid space, we get some formulae for graph phases $(\mathcal{G};\omega,\Lambda)$ by choice $\overline{u}=(x_1,x_2,x_3)$ and $\overline{v}=(y_1,y_2,y_3)$,

$$\omega(\overline{u}) = (x_1,x_2,x_3) \text{ for } \forall\overline{u}\in V(\mathcal{G}),$$

$$\Lambda(\overline{u},\overline{v}) = \frac{x_2y_3-x_3y_2, x_3y_1-x_1y_3, x_1y_2-x_2y_1}{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} \text{ for } \forall(\overline{u},\overline{v})\in E(\mathcal{G}),$$

$$Ca(\mathcal{G};\omega,\Lambda) = \left(\sum_{\overline{u}\in V(\mathcal{G})} x_1(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_2(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_3(\overline{u})\right)$$

and

$$En(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \log(x_1^2(\overline{u})+x_2^2(\overline{u})+x_3^2(\overline{u}).$$

§5.6 RESEARCH PROBLEMS

5.6.1 Besides to embed a graph into k different surfaces $S_1,S_2,\cdots,S_k$ for an integer $k\geq 1$, such as those of discussed in this chapter, we can also consider a graph G embedded in a multi-surface. A multi-surface $\widetilde{S}$ is introduced for characterizing hierarchical structures of topological space. Besides this structure, its base line $L_{\mathscr{B}}$ is common and the same as that of standard surface O_p or N_q. Since all genus of surface in a multi-surface $\widetilde{S}$ is the same, we define the genus $g(\widetilde{S})$ of $\widetilde{S}$ to be the genus of its surface. Define its orientable or non-orientable genus $\widetilde{\gamma}_m^O(G)$, $\widetilde{\gamma}_m^N(G)$ on multi-surface $\widetilde{S}$ consisting of m surfaces S by

$$\widetilde{\gamma}_m^O(G) = \min\left\{ g(\widetilde{S}) \mid G \text{ is } 2-\text{cell embeddable on orinetable multisurface } \widetilde{S}\right\},$$

$$\widetilde{\gamma}_m^N(G) = \min\left\{ g(\widetilde{S}) \mid G \text{ is } 2-\text{cell embeddable on orinetable multisurface } \widetilde{S}\right\}.$$

Then we are easily knowing that $\widetilde{\gamma}_1^O(G) = \gamma(G)$ and $\widetilde{\gamma}_1^N(G) = \widetilde{\gamma}(G)$ by definition. The problems for embedded graphs following are particularly interesting for researchers.

Problem 5.6.1 *Let $n, m \geq 1$ be integers. Determine $\widetilde{\gamma}_m^O(G)$ and $\widetilde{\gamma}_m^N(G)$ for a connected graph G, particularly, the complete graph K_n and the complete bipartite graph $K_{n,m}$.*

Problem 5.6.2 *Let G be a connected graph. Characterize the embedding behavior of G on multi-surface $\widetilde{S}$, particularly, those embeddings whose every facial walk is a circuit, i.e, a strong embedding of G on $\widetilde{S}$.*

The enumeration of non-isomorphic objects is an important problem in combinatorics, particular for maps on surface. See [Liu2] and [Liu4] for details. Similar problems for multi-surface are as follows.

Problem 5.6.3 *Let $\widetilde{S}$ be a multi-surface. Enumerate embeddings or maps on $\widetilde{S}$ by parameters, such as those of order, size, valency of rooted vertex or rooted face, $\cdots$.*

Problem 5.6.4 *Enumerate embeddings on multi-surfaces for a connected graph G.*

For a connected graph G, its orientable, non-orientable genus polynomial $g_m[G](x)$, $\widetilde{g}_m[G](x)$ is defined to be

$$g_m[G](x) = \sum_{i\geq 0} g_{mi}^O(G)x^i \quad \text{and} \quad \widetilde{g}_m[G](x) = \sum_{i\geq 0} g_{mi}^N(G)x^i,$$

where $g_{mi}^O(G)$, $g_{mi}^N(G)$ are the numbers of G on orientable or non-orientable multi-surface $\widetilde{S}$ consisting of m surfaces of genus i.

Problem 5.6.5 *Let $m \geq 1$ be an integer. Determine $g_m[G](x)$ and $\widetilde{g}_m[G](x)$ for a connected graph G, particularly, for the complete or complete bipartite graph, the cube, the ladder, the bouquet, $\cdots$.*

5.6.2 A graphical property $P(G)$ is called to be *subgraph hereditary* if for any subgraph $H \subseteq G$, H posses $P(G)$ whenever G posses the property $P(G)$. For example, the properties: *G is complete* and *the vertex coloring number $\chi(G) \leq k$* both are subgraph hereditary. The hereditary property of a graph can be generalized by the following way.

Finding the behavior of a graph in space is an interesting, also important objective for application. There are many open problems on this objective connecting with classical mathematics. Let G and H be two graphs in a space $\widetilde{M}$. If there is a smooth mapping ς in $C(\widetilde{M})$ such that $\varsigma(G) = H$, then we say G and H are *equivalent in $\widetilde{M}$*. Many conceptions in graph theory can be included in this definition, such as *graph homomorphism, graph equivalent*, $\cdots$, etc.

Problem 5.6.6 *Applying different smooth mappings in a space such as smooth mappings in* $\mathbf{R}^3$ *or* $\mathbf{R}^4$ *to classify graphs and to find their invariants.*

Problem 5.6.7 *Find which parameters already known in graph theory for a graph is invariant or to find the smooth mapping in a space on which this parameter is invariant.*

Problem 5.6.8 *Find which parameters for a graph can be used to a graph in a space. Determine combinatorial properties of a graph in a space.*

Consider a graph in a Euclid space of dimension 3. All of its edges are seen as a structural member, such as steel bars or rods and its vertices are hinged points. Then we raise the following problem.

Problem 5.6.9 *Applying structural mechanics to classify what kind of graph structures are stable or unstable. Whether can we discover structural mechanics of dimension*≥ 4 *by this idea?*

We have known the orbit of a point under an action of a group, for example, a torus is an orbit of $Z \times Z$ action on a point in $\mathbf{R}^3$. Similarly, we can also define an *orbit of a graph in a space* under an action on this space.

Let $\mathcal{G}$ *be a graph in a multi-space* $\widetilde{M}$ *and* Π *a family of actions on* $\widetilde{M}$*. Define an orbit* $Or(\mathcal{G})$ *by*

$$Or(\mathcal{G}) = \{\pi(\mathcal{G}) | \ \forall \pi \in \Pi\}.$$

Problem 5.6.10 *Given an action* π*, continuous or discontinuous on a space* $\widetilde{M}$*, for example* $\mathbf{R}^3$ *and a graph* $\mathcal{G}$ *in* $\widetilde{M}$*, find the orbit of* $\mathcal{G}$ *under the action of* π*. When can we get a closed geometrical object by this action?*

Problem 5.6.11 *Given a family* $\mathcal{A}$ *of actions, continuous or discontinuous on a space* $\widetilde{M}$ *and a graph* $\mathcal{G}$ *in* $\widetilde{M}$*, find the orbit of* $\mathcal{G}$ *under these actions in* $\mathcal{A}$*. Find the orbit of a vertex or an edge of* $\mathcal{G}$ *under the action of* $\mathcal{G}$*, and when are they closed?*

5.6.3 There is an alternative way for defining transformable graph phases, i.e., by homotopy groups in a topological space stated as follows:

Let $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ be two graph phases. If there is a continuous mapping $H : C\left(\widetilde{M}\right) \times I \to C\left(\widetilde{M}\right) \times I$, $I = [0, 1]$ such that $H\left(C\left(\widetilde{M}\right), 0\right) = (\mathcal{G}_1; \omega_1, \Lambda_1)$ and $H\left(C\left(\widetilde{M}\right), 1\right) = (\mathcal{G}_2; \omega_2, \Lambda_2)$, then $(\mathcal{G}_1; \omega_1, \Lambda_1)$ and $(\mathcal{G}_2; \omega_2, \Lambda_2)$ are said two *transformable graph phases*.

Similar to topology, we can also introduce product on homotopy equivalence classes and prove that all homotopy equivalence classes form a group. This group is called a *fundamental group* and denote it by $\pi(\mathcal{G};\omega,\Lambda)$. In topology there is a famous theorem, called the *Seifert and Van Kampen theorem* for characterizing fundamental groups $\pi_1(\mathcal{A})$ of topological spaces $\mathcal{A}$ restated as follows (See [Sti1] for details).

Suppose $\mathcal{E}$ is a space which can be expressed as the union of path-connected open sets $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A}\bigcap\mathcal{B}$ is path-connected and $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ have respective presentations

$$\langle a_1,\cdots,a_m;r_1,\cdots,r_n\rangle,\qquad \langle b_1,\cdots,b_m;s_1,\cdots,s_n\rangle,$$

while $\pi_1(\mathcal{A}\bigcap\mathcal{B})$ is finitely generated. Then $\pi_1(\mathcal{E})$ has a presentation

$$\langle a_1,\cdots,a_m,b_1,\cdots,b_m;r_1,\cdots,r_n,s_1,\cdots,s_n,u_1=v_1,\cdots,u_t=v_t\rangle,$$

where $u_i,v_i,i=1,\cdots,t$ are expressions for the generators of $\pi_1(\mathcal{A}\bigcap\mathcal{B})$ in terms of the generators of $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ respectively.

Similarly, there is a problem for the fundamental group $\pi(\mathcal{G};\omega,\Lambda)$ of a graph phase $(\mathcal{G};\omega,\Lambda)$ following.

Problem 5.6.12 *Find results similar to that of Seifert and Van Kampen theorem for the fundamental group of a graph phase and characterize it.*

5.6.4 In Euclid space $\mathbf{R}^n$, an *n-ball of radius r* is determined by

$$B^n(r)=\{(x_1,x_2,\cdots,x_n)|x_1^2+x_2^2+\cdots+x_n^2\leq r\}.$$

Now we choose m n-balls $B_1^n(r_1),B_2^n(r_2),\cdots,B_m^n(r_m)$, where for any integers $i,j,1\leq i,j\leq m$, $B_i^n(r_i)\bigcap B_j^n(r_j)=$ or not and $r_i=r_j$ or not. An *n-multi-ball* is a union

$$\widetilde{B}=\bigcup_{k=1}^{m}B_k^n(r_k).$$

Then an *n-multi-manifold* is a Hausdorff space with each point in this space has a neighborhood homeomorphic to an n-multi-ball.

Problem 5.6.13 *For an integer $n,n\geq 2$, classifies n-multi-manifolds. Especially, classifies* 2*-multi-manifolds.*

CHAPTER 6.

Map Geometry

A Smarandache geometry is nothing but a Smarandcahe multi-space consisting of just two geometrical spaces A_1 and A_2, associated with an axiom L such that L holds in A_1 but not holds in A_2, or only hold not in both A_1 and A_2 but in distinct ways, a miniature of multi-space introduced by Smarandache in 1969. The points in such a geometry can be classified into three classes, i.e., elliptic, Euclidean and hyperbolic types. For the case only with finite points of elliptic and hyperbolic types, such a geometry can be characterized by combinatorial map. Thus is the geometry on Smarandache manifolds of dimension 2, i.e., map geometry. We introduce Smarandache geometry including paradoxist geometry, non-geometry, counter-projective geometry, anti-geometry and Iseri's s-manifolds in Section 6.1. These map geometry with or without boundary are discussed and paradoxist geometry, non-geometry, counter-projective geometry and anti-geometry are constructed on such map geometry in Sections 6.2 and 6.3. The curvature of an s-line is defined in Section 6.4, where a condition for a map on map geometry (M, μ) being Smarandachely is found. Section 6.5 presents the enumeration result for non-equivalent map geometries underlying a simple graph Γ. All of these decision consist the fundamental of the following chapters.

§6.1 SMARANDACHE GEOMETRY

6.1.1 Geometrical Introspection. As we known, mathematics is a powerful tool of sciences for its unity and neatness, without any shade of mankind. On the other hand, it is also a kind of aesthetics deep down in one's mind. There is a famous proverb says that *only the beautiful things can be handed down to today*, which is also true for the mathematics.

Here, the terms *unity* and *neatness* are relative and local, maybe also have various conditions. For obtaining a good result, many unimportant matters are abandoned in the research process. Whether are those matters still unimportant in another time? It is not true. That is why we need to think a queer question: *what are lost in the classical mathematics?*

For example, a compact surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along its a given direction ([Mas1] or [Sti1]). If label each pair of edges by a letter $e, e \in \mathcal{E}$, a surface S is also identified to a cyclic permutation such that each edge $e, e \in \mathcal{E}$ just appears two times in S, one is e and another is e^{-1} (orientable) or e (non-orientable). Let $a, b, c, \cdots$ denote letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in a linear order on a surface S (or a string of letters on S). Then, an orientable surface can be represented by

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where, $a \in \mathcal{E}$ and A, B, C denote strings of letter. Three elementary transformations are defined as follows:

(O_1) $(A, a, a^{-1}, B) \Leftrightarrow (A, B)$;

(O_2) (i) $(A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1})$;

(ii) $(A, a, b, B, a, b) \Leftrightarrow (A, c, B, c)$;

(O_3) (i) $(A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C)$;

(ii) $(A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1})$.

If a surface S_0 can be obtained by these elementary transformations O_1-O_3 from a surface S, it is said that S is *elementary equivalent* with S_0, denoted by $S \sim_{El} S_0$.

We have known the following formulae from [Liu1]:

(1) $(A, a, B, b, C, a^{-1}, D, b^{-1}, E) \sim_{El} (A, D, C, B, E, a, b, a^{-1}, b^{-1})$;

(2) $(A, c, B, c) \sim_{El} (A, B^{-1}, C, c, c)$;

(3) $(A, c, c, a, b, a^{-1}, b^{-1}) \sim_{El} (A, c, c, a, a, b, b)$.

Then we can get the classification theorem of compact surfaces as follows [Mas1]:

Any compact surface is homeomorphic to one of the following standard surfaces:

(P_0) The sphere: aa^{-1};

(P_n) The connected sum of $n, n \geq 1$, tori:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1};$$

(Q_n) The connected sum of $n, n \geq 1$, projective planes:

$$a_1a_1a_2a_2\cdots a_na_n.$$

As we have discussed in Chapter 2, a combinatorial map is just a kind of decomposition of a surface. Notice that all the standard surfaces are one face map underlying an one vertex graph, i.e., a bouquet B_n with $n \geq 1$. By a combinatorial view, *a combinatorial map is nothing but a surface*. This assertion is needed clarifying. For example, let us see the left graph Π_4 in Fig.3.1.1, which is a tetrahedron.

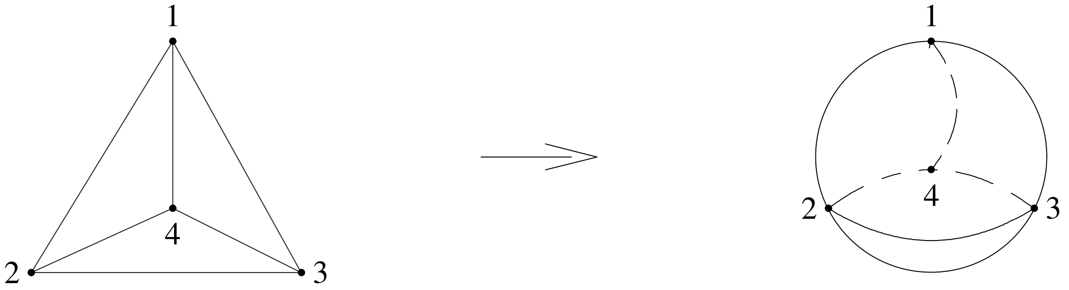


Fig.6.1.1

Whether can we say Π_4 is a sphere? Certainly NOT. Since any point u on a sphere has a neighborhood $N(u)$ homeomorphic to an open disc, thereby all angles incident with the point 1 must be 120° degree on a sphere. But in Π_4, those are only 60° degree. For making them same in a topological sense, i.e., homeomorphism, we must blow up the Π_4 and make it become a sphere. This physical processing is shown in the Fig.3.1. Whence, for getting the classification theorem of compact surfaces, we lose the *angle,area, volume,distance,curvature,*$\cdots$, etc. which are also lost in combinatorial maps.

By geometrical view, the *Klein Erlanger Program* says that *any geometry is nothing but find invariants under a transformation group of this geometry*. This is essentially the group action idea and widely used in mathematics today. Surveying topics appearing in publications for combinatorial maps, we know the following problems are applications of *Klein Erlanger Program*:

(1) *to determine isomorphism maps or rooted maps;*
(2) *to determine equivalent embeddings of a graph;*
(3) *to determine an embedding whether exists or not;*
(4) *to enumerate maps or rooted maps on a surface;*
(5) *to enumerate embeddings of a graph on a surface;*
(6) $\cdots$, etc.

All the problems are extensively investigated by researches in the last century and papers related those problems are still frequently appearing in journals today. Then,

what are their importance to classical mathematics?

and

what are their contributions to sciences?

Today, we have found that combinatorial maps can contribute an underlying frame for applying mathematics to sciences, i.e., through by map geometries or by graphs in spaces.

6.1.2 Smarandache Geometry. The *Smarandache geometry* was proposed by Smarandache [Sma1] in 1969, which is a generalization of classical geometries, i.e., these *Euclid, Lobachevshy-Bolyai-Gauss* and *Riemann geometries* may be united altogether in a same space, by some Smarandache geometries. Such geometry can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. Smarandache geometries are also connected with the *Relativity Theory* because they include Riemann geometry in a subspace and with the *Parallel Universes* because they combine separate spaces into one space too. For a detail illustration, we need to consider classical geometry first.

As we known, the axiom system of *Euclid geometry* consists of 5 axioms following:

(A1) *There is a straight line between any two points.*
(A2) *A finite straight line can produce a infinite straight line continuously.*
(A3) *Any point and a distance can describe a circle.*

(A4) *All right angles are equal to one another.*

(A5) *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

The axiom (A5) can be also replaced by:

(A5') *given a line and a point exterior this line, there is one line parallel to this line.*

The *Lobachevshy-Bolyai-Gauss geometry*, also called *hyperbolic geometry*, is a geometry with axioms $(A1)-(A4)$ and the following axiom $(L5)$:

(L5) *there are infinitely many lines parallel to a given line passing through an exterior point.*

and the *Riemann geometry*, also called *elliptic geometry*, is a geometry with axioms $(A1)-(A4)$ and the following axiom $(R5)$:

there is no parallel to a given line passing through an exterior point.

By a thought of anti-mathematics: *not in a nihilistic way, but in a positive one, i.e., banish the old concepts by some new ones: their opposites*, Smarandache [Sma1] introduced the *paradoxist geometry, non-geometry, counter-projective geometry* and *anti-geometry* by contradicts respectively to axioms $(A1)-(A5)$ in Euclid geometry following.

Paradoxist Geometry. In this geometry, its axioms consist of $(A1)-(A4)$ and one of the following as the axiom $(P5)$:

(1) *There are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(2) *There are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(3) *There are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $l_1, l_2, \cdots, l_k, k \geq 2$ *pass through the point and do not intersect the initial line.*

(4) *There are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(5) *There are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

Non-Geometry. The non-geometry is a geometry by denial some axioms of $(A1)-(A5)$, such as:

$(A1^-)$ *It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

$(A2^-)$ *It is not always possible to extend by continuity a finite line to an infinite line.*

$(A3^-)$ *It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

$(A4^-)$ *Not all the right angles are congruent.*

$(A5^-)$ *If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

Counter-Projective Geometry. Denoted by P the point set, L the line set and R a relation included in $P\times L$. A counter-projective geometry is a geometry with these counter-axioms $(C_1)-(C_3)$:

$(C1)$ *there exist: either at least two lines, or no line, that contains two given distinct points.*

$(C2)$ *let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing p_1, p_2 and the line containing q_1, q_2 do not intersect.*

$(C3)$ *every line contains at most two distinct points.*

Anti-Geometry. A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry. As shown in [KuA1], there are at least $2^{21}-1$ such anti-geometries.

In general, a Smarandache geometry is defined as follows.

Definition 6.1.1 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969).

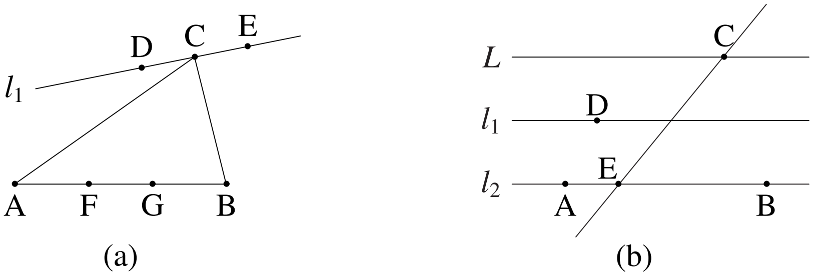
In the Smarandache geometry, points, lines, planes, spaces, triangles, $\cdots$, etc are called s-points, s-lines, s-planes, s-spaces, s-triangles, $\cdots$, respectively in order to distinguish them from classical geometries.

An example of Smarandache geometry in the classical geometrical sense is shown in the following.

Example 6.1.1 Let us consider a Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C. Then such a geometry is a Smarandache geometry by the following observations.

Observation 1. The axiom (E1) that through any two distinct points there exist one line passing through them is now replaced by: *one s-line* and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.9.1.1(a).

Observation 2. The axiom (E5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*. Let L be an s-line passes through C and is parallel in the Euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.9.1.1(b).



(a) (b)

Fig.6.1.1

6.1.3 Smarandache Manifold. A *Smarandache manifold* is an n-dimensional manifold that support a Smarandache geometry. For $n = 2$, a nice model for Smarandache geometries called *s-manifolds* was found by Iseri in [Ise1]-[Ise2] defined as follows:

An s-manifold is any collection $C(T, n)$ of these equilateral triangular disks $T_i, 1 \le i \le n$ satisfying the following conditions:

(i) *each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks T_i, T_j, $1 \leq i, j \leq n$ and $i \neq j$;*

(ii) *each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.*

The vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called an *elliptic vertex*, a *Euclidean vertex* or a *hyperbolic vertex*, respectively.

In the plane, an elliptic vertex O, a Euclidean vertex P and a hyperbolic vertex Q and an s-line L_1, L_2 or L_3 passes through points O, P or Q are shown in Fig.6.1.2(a), (b), (c), respectively.

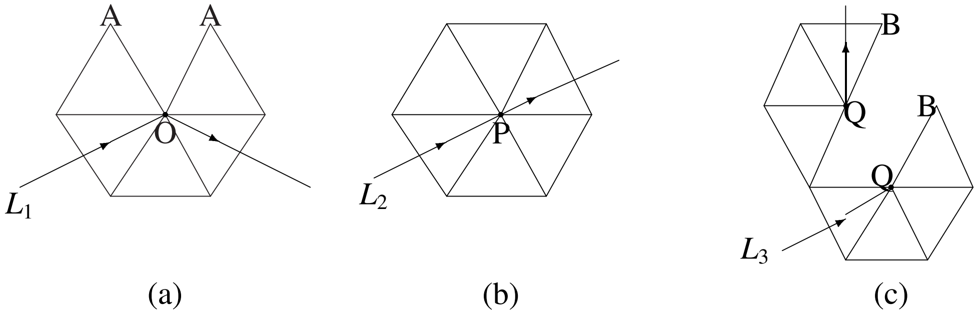


Fig.6.1.2

Smarandache paradoxist geometries and non-geometries can be realized by s-manifolds, but other Smarandache geometries can be only partly realized by this kind of manifolds. Readers are inferred to Iseri's book [Ise1] for those geometries.

An s-manifold is called closed if each edge is shared exactly by two triangular disks. An elementary classification for closed s-manifolds by planar triangulation were introduced in [Mao10]. They are classified into 7 classes. Each of those classes is defined in the following.

Classical Type:

(1) $\Delta_1 = \{5 - \text{regular planar triangular maps}\}$ (*elliptic*);

(2) $\Delta_2 = \{6 - \text{regular planar triangular maps}\}$(*euclidean*);

(3) $\Delta_3 = \{7 - \text{regular planar triangular maps}\}$(*hyperbolic*).

Smarandache Type:

(4) $\Delta_4 = \{\text{planar triangular maps with vertex valency 5 and 6}\}$ (*euclid-elliptic*);

(5) Δ_5 = {planar triangular maps with vertex valency 5 and 7} (*elliptic-hyperbolic*);

(6) Δ_6 = {planar triangular maps with vertex valency 6 and 7} (*euclid-hyperbolic*);

(7) Δ_7 = {planar triangular maps with vertex valency 5, 6 and 7} (*mixed*).

It is proved in [Mao10] that $|\Delta_1| = 2$, $|\Delta_5| \geq 2$ and $|\Delta_i|, i = 2,3,4,6,7$ are infinite (See also [Mao37] for details). Iseri proposed a question in [Ise1]: *Do the other closed 2-manifolds correspond to s-manifolds with only hyperbolic vertices?* Since there are infinite Hurwitz maps, i.e., $|\Delta_3|$ is infinite, the answer is affirmative.

§6.2 MAP GEOMETRY WITHOUT BOUNDARY

6.2.1 Map Geometry Without Boundary. A combinatorial map M can be also used for a model of constructing Smarandache geometry. By a geometrical view, this model is a generalizations of Isier's model for Smarandache geometry. For a given map on a locally orientable surface, map geometries without boundary are defined in the following definition.

Definition 6.2.1 *For a combinatorial map M with each vertex valency≥ 3, associates a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry without boundary, $\mu(u)$ an angle factor of the vertex u and orientablle or non-orientable if M is orientable or not.*

A vertex $u \in V(M)$ with $\rho_M(u)\mu(u) < 2\pi$, $= 2\pi$ or $> 2\pi$ can be realized in a Euclidean space $\mathbf{R}^3$, such as those shown in Fig.6.2.1, respectively.

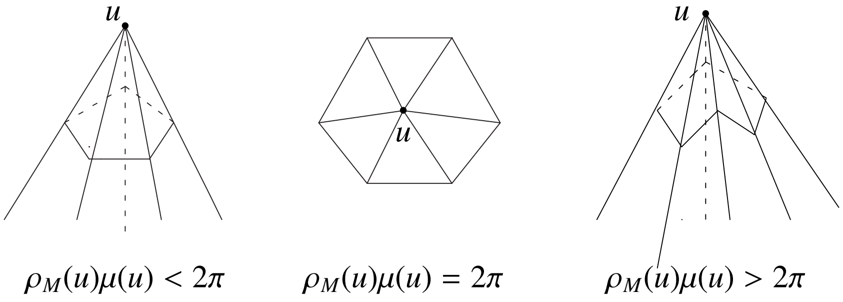


Fig.6.2.1

As we have pointed out in Section 6.1, this kind of realization is not a surface, but it is homeomorphic to a locally orientable surface by a view of topological equivalence. Similar to s-manifolds, we also classify points in a map geometry (M, μ) without boundary into *elliptic points*, *Euclidean points* and *hyperbolic points*, defined in the next definition.

Definition 6.2.2 *A point u in a map geometry (M, μ) is said to be elliptic, Euclidean or hyperbolic if $\rho_M(u)\mu(u) < 2\pi$, $\rho_M(u)\mu(u) = 2\pi$ or $\rho_M(u)\mu(u) > 2\pi$.*

Then we get the following results.

Theorem 6.2.1 *Let M be a map with $\rho_M(v) \geq 3$ for $\forall v \in V(M)$. Then for $\forall u \in V(M)$, there is a map geometry (M, μ) without boundary such that u is elliptic, Euclidean or hyperbolic.*

Proof Since $\rho_M(u) \geq 3$, we can choose an angle factor $\mu(u)$ such that $\mu(u)\rho_M(u) < 2\pi$, $\mu(u)\rho_M(u) = 2\pi$ or $\mu(u)\rho_M(u) > 2\pi$. Notice that

$$0 < \frac{2\pi}{\rho_M(u)} < \frac{4\pi}{\rho_M(u)}.$$

Thereby we can always choose $\mu(u)$ satisfying that $0 < \mu(u) < \dfrac{4\pi}{\rho_M(u)}$. □

Theorem 6.2.2 *Let M be a map of order≥ 3 and $\rho_M(v) \geq 3$ for $\forall v \in V(M)$,. Then there exists a map geometry (M, μ) without boundary in which elliptic, Euclidean and hyperbolic points appear simultaneously.*

Proof According to Theorem 6.2.1, we can always choose an angle factor μ such that a vertex $u, u \in V(M)$ to be elliptic, or Euclidean, or hyperbolic. Since $|V(M)| \geq 3$, we can even choose the angle factor μ such that any two different vertices $v, w \in V(M)\backslash\{u\}$ to be elliptic, or Euclidean, or hyperbolic as we wish. Then the map geometry (M, μ) makes the assertion hold. □

A *geodesic* in a manifold is a curve as straight as possible. Applying conceptions such as angles and straight lines in a Euclid geometry, we define s-lines and s-points in a map geometry in the next definition.

Definition 6.2.3 *Let (M, μ) be a map geometry without boundary and let $S(M)$ be the locally orientable surface represented by a plane polygon on which M is embedded. A point P on $S(M)$ is called an s-point. A line L on $S(M)$ is called an s-line if it is straight in each face of M and each angle on L has measure $\dfrac{\rho_M(v)\mu(v)}{2}$ when it passes through a vertex v on M.*

Two examples for s-lines on the torus are shown in the Fig.6.2.2(a) and (b), where $M = M(B_2)$, $\mu(u) = \frac{\pi}{2}$ for the vertex u in (a) and

$$\mu(u) = \frac{135 - \arctan(2)}{360}\pi$$

for the vertex u in (b), i.e., u is Euclidean in (a) but elliptic in (b). Notice that in (b), the s-line L_2 is self-intersected.

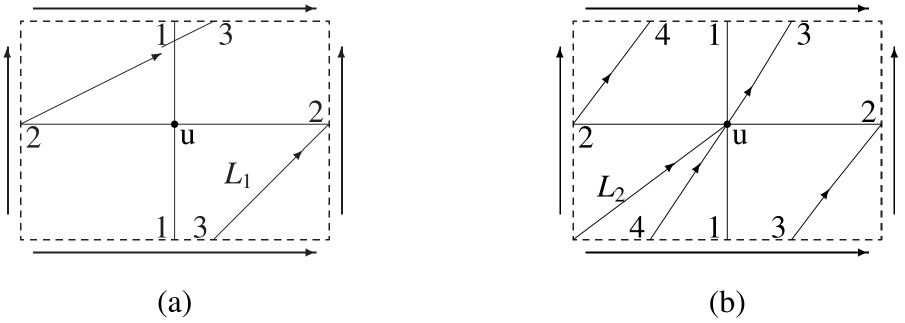
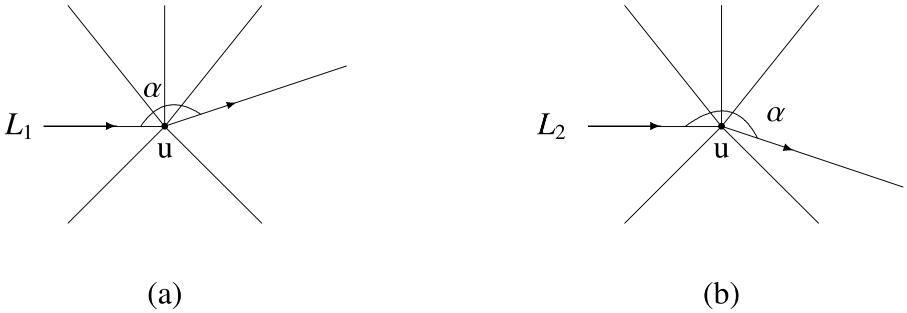


Fig.6.2.2

If an s-line passes through an elliptic point or a hyperbolic point u, it must has an angle $\frac{\mu(u)\rho_M(u)}{2}$ with the entering line, not 180° which are explained in Fig.6.2.3.



$$\alpha = \frac{\mu(u)\rho_M(u)}{2} < \pi \qquad\qquad \alpha = \frac{\mu(u)\rho_M(u)}{2} > \pi$$

Fig.6.2.3

6.2.2 Paradoxist Map Geometry. In the Euclid geometry, a right angle is an angle with measure $\frac{\pi}{2}$, half of a straight angle and parallel lines are straight lines never intersecting. They are very important research objects. Many theorems characterize properties of them in classical Euclid geometry. Similarly, in a map geometry, we can also define a straight angle, a right angle and parallel s-lines by Definition 6.2.2. Now a *straight angle* is an angle with measure π for points not being vertices of M and $\frac{\rho_M(u)\mu(u)}{2}$ for $\forall u \in V(M)$. A *right angle* is an angle with a half measure of a straight angle. Two s-lines are said *parallel* if they are never intersecting. The following result asserts that there exists map paradoxist geometry without boundary.

Theorem 6.2.3 *Let M be a map on a locally orientable surface with $|M| \geq 3$ and $\rho_M(u) \geq 3$ for $\forall u \in V(M)$. Then there exists an angle factor $\mu : V(M) \to [0, 4\pi)$ such that (M, μ) is a Smarandache geometry by denial the axiom (E5) with axioms (E5),(L5) and (R5).*

Proof By the assumption $\rho_M(u) \geq 3$, we can always choose an angle factor μ such that $\mu(u)\rho_M(u) < 2\pi$, $\mu(v)\rho_M(u) = 2\pi$ or $\mu(w)\rho_M(u) > 2\pi$ for three vertices $u, v, w \in V(M)$, i.e., there elliptic, or Euclidean, or hyperbolic points exist in (M, μ) simultaneously. The proof is divided into three cases.

Case 1. M is a planar map.

Choose L being a line under the map M, not intersection with it, $u \in (M, \mu)$. Then if u is Euclidean, there is one and only one line passing through u not intersecting with L, and if u is elliptic, there are infinite many lines passing through u not intersecting with L, but if u is hyperbolic, then each line passing through u will intersect with L. See for example, Fig.6.2.4 in where the planar graph is a complete graph K_4 on a sphere and points $1, 2$ are elliptic, 3 is Euclidean but the point 4 is hyperbolic. Then all lines in the field A do not intersect with L, but each line passing through the point 4 will intersect with the line L. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (E5) with these axioms (E5), (L5) and (R5).

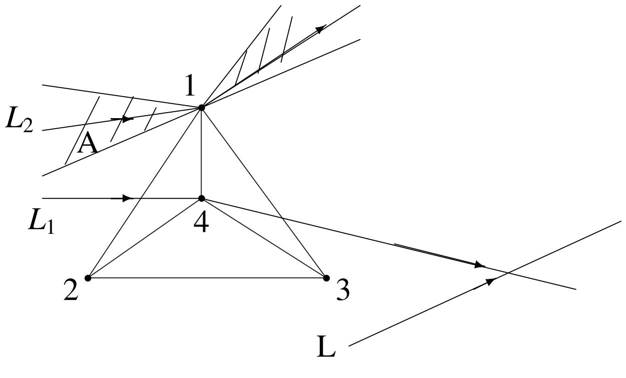


Fig.6.2.4

Case 2. M is an orientable map.

According to the classifying theorem of surfaces, We only need to prove this assertion on a torus. In this case, lines on a torus has the following property (see [NiS1] for details):

if the slope ς *of a line L is a rational number, then L is a closed line on the torus. Otherwise, L is infinite, and moreover L passes arbitrarily close to every point on the torus.*

Whence, if L_1 is a line on a torus with an irrational slope not passing through an elliptic or a hyperbolic point, then for any point u exterior to L_1, if u is a Euclidean point, then there is only one line passing through u not intersecting with L_1, and if u is elliptic or hyperbolic, any s-line passing through u will intersect with L_1.

Now let L_2 be a line on the torus with a rational slope not passing through an elliptic or a hyperbolic point, such as the the line L_2 shown in Fig.6.2.5, in where v is a Euclidean point. If u is a Euclidean point, then each line L passing through u with rational slope in the area A will not intersect with L_2, but each line passing through u with irrational slope in the area A will intersect with L_2.

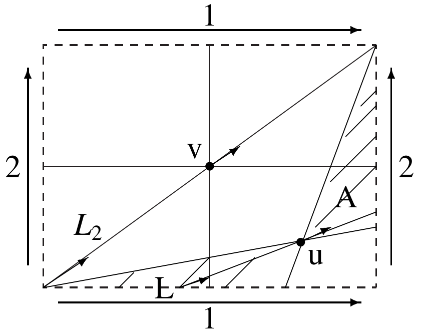
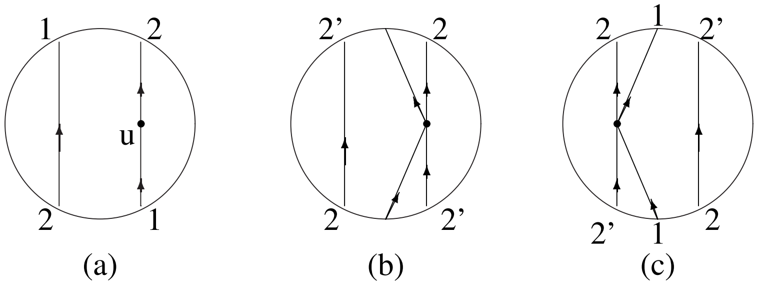


Fig.6.2.5

Therefore, (M, μ) is a Smarandache geometry by denial the axiom (E5) with axioms (E5), (L5) and (R5) in the orientable case.

Case 3. M is a non-orientable map.

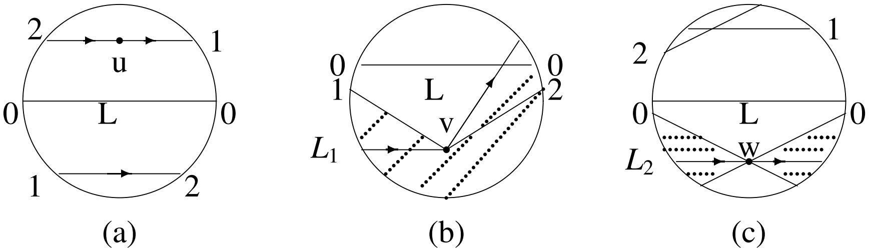
Similar to Case 2, we only need to prove this result for the projective plane. A line in a projective plane is shown in Fig.6.2.6(a), (b) or (c), in where case (a) is a line passing through a Euclidean point, (b) passing through an elliptic point and (c) passing through a hyperbolic point.



(a) (b) (c)

Fig.6.2.6

Let L be a line passing through the center of the circle. Then if u is a Euclidean point, there is only one line passing through u such as the case (a) in Fig.6.2.7. If v is an elliptic point then there is an s-line passing through it and intersecting with L such as the case (b) in Fig.6.2.7. We assume the point 1 is a point such that there exists a line passing through 1 and 0, then any line in the shade of Fig.6.2.7(b) passing through v will intersect with L.



(a) (b) (c)

Fig.6.2.7

If w is a Euclidean point and there is a line passing through it not intersecting with L such as the case (c) in Fig.6.2.7, then any line in the shade of Fig.6.2.7(c) passing through w will not intersect with L. Since the position of the vertices of a map M on a projective plane can be choose as our wish, we know (M, μ) is a Smarandache geometry by denial the axiom (E5) with axioms (E5),(L5) and (R5).

Combining discussions of Cases 1, 2 and 3, the proof is complete. □

6.2.3 Map Non-Geometry. Similar to those of Iseri's s-manifolds, there are non-geometry, anti-geometry and counter-projective geometry, $\cdots$ in map geometry without boundary.

Theorem 6.2.4 *There exists a non-geometry in map geometries without boundary.*

Proof We prove there are map geometries without boundary satisfying axioms (A_1^-)–(A_5^-). Let (M,μ) be such a map geometry with elliptic or hyperbolic points.

(1) Assume u is a Euclidean point and v is an elliptic or hyperbolic point on (M,μ). Let L be an s-line passing through points u and v in a Euclid plane. Choose a point w in L after but nearly enough to v when we travel on L from u to v. Then there does not exist a line from u to w in the map geometry (M,μ) since v is an elliptic or hyperbolic point. So the axiom (A_1^-) is true in (M,μ).

(2) In a map geometry (M,μ), an s-line maybe closed such as we have illustrated in the proof of Theorem 6.2.3. Choose any two points A, B on a closed s-line L in a map geometry. Then the s-line between A and B can not continuously extend to indefinite in (M,μ). Whence the axiom (A_2^-) is true in (M,μ).

(3) An m-circle in a map geometry is defined to be a set of continuous points in which all points have a given distance to a given point. Let C be a m-circle in a Euclid plane. Choose an elliptic or a hyperbolic point A on C which enables us to get a map geometry (M,μ). Then C has a gap in A by definition of an elliptic or hyperbolic point. So the axiom (A_3^-) is true in a map geometry without boundary.

(4) By the definition of a right angle, we know that a right angle on an elliptic point can not equal to a right angle on a hyperbolic point. So the axiom (A_4^-) is held in a map geometry with elliptic or hyperbolic points.

(5) The axiom (A_5^-) is true by Theorem 6.2.3.

Combining these discussions of (i)-(v), we know that there are non-geometries in map geometries. This completes the proof. □

6.2.4 Map Anti-Geometry. The *Hilbert's axiom system* for a Euclid plane geometry consists five group axioms stated in the following, where we denote each group by a capital *Roman* numeral.

I. Incidence

$I-1$. *For every two points A and B, there exists a line L that contains each of the points A and B.*

$I-2$. *For every two points A and B, there exists no more than one line that contains each of the points A and B.*

$I-3$. *There are at least two points on a line. There are at least three points not on a line.*

II. Betweenness

$II-1$. *If a point B lies between points A and C, then the points A, B and C are distinct points of a line, and B also lies between C and A.*

$II-2$. *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*

$II-3$. *Of any three points on a line, there exists no more than one that lies between the other two.*

$II-4$. *Let A, B and C be three points that do not lie on a line, and let L be a line which does not meet any of the points A, B and C. If the line L passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.*

III. Congruence

$III-1$. *If A_1 and B_1 are two points on a line L_1, and A_2 is a point on a line L_2 then it is always possible to find a point B_2 on a given side of the line L_2 through A_2 such that the segment A_1B_1 is congruent to the segment A_2B_2.*

$III-2$. *If a segment A_1B_1 and a segment A_2B_2 are congruent to the segment AB, then the segment A_1B_1 is also congruent to the segment A_2B_2.*

$III-3$. *On the line L, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line L_1, let A_1B_1 and B_1C_1 be two segments, which except for B_1 also have no point in common. In that case, if AB is congruent to A_1B_1 and BC is congruent to B_1C_1, then AC is congruent to A_1C_1.*

$III-4$. *Every angle can be copied on a given side of a given ray in a uniquely determined way.*

$III-5$ *If for two triangles ABC and $A_1B_1C_1$, AB is congruent to A_1B_1, AC is congruent to A_1C_1 and $\angle BAC$ is congruent to $\angle B_1A_1C_1$, then $\angle ABC$ is congruent to $\angle A_1B_1C_1$.*

IV. Parallels

$IV-1$. *There is at most one line passes through a point P exterior a line L that is parallel to L.*

V. Continuity

$V-1$(Archimedes) *Let AB and CD be two line segments with $|AB| \geq |CD|$. Then* *there is an integer m such that*

$$m|CD| \leq |AB| \leq (m+1)|CD|.$$

$V-2$(Cantor) Let $A_1B_1, A_2B_2, \cdots, A_nB_n, \cdots$ be a segment sequence on a line L. If

$$A_1B_1 \supseteq A_2B_2 \supseteq \cdots \supseteq A_nB_n \supseteq \cdots,$$

then there exists a common point X on each line segment A_nB_n *for any integer* $n, n \geq 1$.

Smarandache defined an anti-geometry by denial some axioms in Hilbert axiom system for Euclid geometry. Similar to the discussion in the reference [Ise1], We obtain the following result for anti-geometry in map geometry without boundary.

Theorem 6.2.5 *Unless axioms* $I-3$, $II-3$, $III-2$, $V-1$ *and* $V-2$, *an anti-geometry can be gotten from map geometry without boundary by denial other axioms in Hilbert axiom system.*

Proof The axiom $I-1$ has been denied in the proof of Theorem 6.2.4. Since there maybe exists more than one line passing through two points A and B in a map geometry with elliptic or hyperbolic points u such as those shown in Fig.6.2.8. So the axiom $II-2$ can be Smarandachely denied.

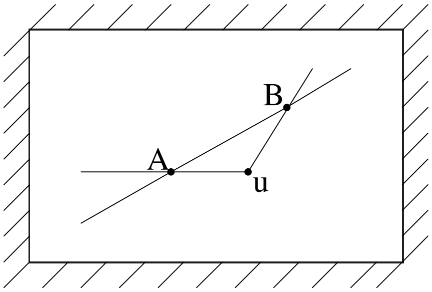


Fig.6.2.8

Notice that an s-line maybe has self-intersection points in a map geometry without boundary. So the axiom $II-1$ can be denied. By the proof of Theorem 6.2.4, we know that for two points A and B, an s-line passing through A and B may not exist. Whence, the axiom $II-2$ can be denied. For the axiom $II-4$, see Fig.6.2.9, in where v is a non-Euclidean point such that $\rho_M(v)\mu(v) \geq 2(\pi + \angle ACB)$ in a map geometry.

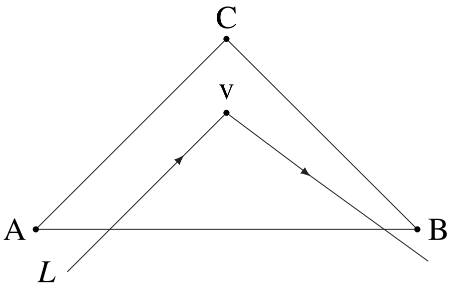


Fig.6.2.9

So $II - 4$ can be also denied. Notice that an s-line maybe has self-intersection points. There are maybe more than one s-lines passing through two given points A, B. Therefore, the axioms $III - 1$ and $III - 3$ are deniable. For denial the axiom $III - 4$, since an elliptic point u can be measured at most by a number $\frac{\rho_M(u)\mu(u)}{2} < \pi$, i.e., there is a limitation for an elliptic point u. Whence, an angle with measure bigger than $\frac{\rho_M(u)\mu(u)}{2}$ can not be copied on an elliptic point on a given ray.

Because there are maybe more than one s-lines passing through two given points A and B in a map geometry without boundary, the axiom $III - 5$ can be Smarandachely denied in general such as those shown in Fig.6.2.10(a) and (b) where u is an elliptic point.

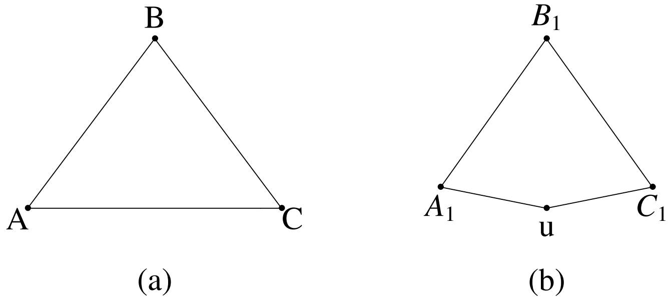


Fig.6.2.10

For the parallel axiom $IV - 1$, it has been denied by the proofs of Theorems 6.2.3 and 6.2.4.

Notice that axioms $I - 3$, $II - 3$ $III - 2$, $V - 1$ and $V - 2$ can not be denied in a map geometry without boundary. This completes the proof. □

6.2.5 Counter-Projective Map Geometry. For counter-projective geometry, we know a result following.

Theorem 6.2.6 *Unless the axiom* ($C3$), *a counter-projective geometry can be gotten from map geometry without boundary by denial axioms* ($C1$) *and* ($C2$).

Proof Notice that axioms ($C1$) and ($C2$) have been denied in the proof of Theorem 6.2.5. Since a map is embedded on a locally orientable surface, every s-line in a map geometry without boundary may contains infinite points. Therefore the axiom ($C3$) can not be Smarandachely denied. □

§6.3 MAP GEOMETRY WITH BOUNDARY

6.3.1 Map Geometry With Boundary. A *Poincaré's model* for a hyperbolic geometry is an upper half-plane in which lines are upper half-circles with center on the x-axis or upper straight lines perpendicular to the x-axis such as those shown in Fig.6.3.1.

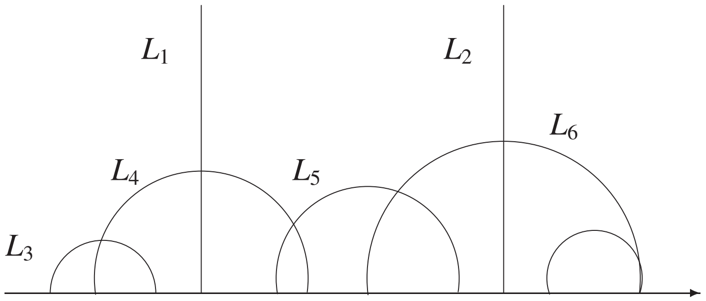


Fig.6.3.1

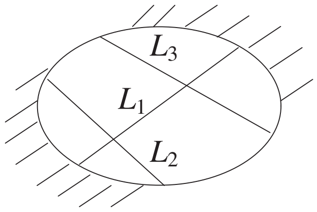


Fig.6.3.2

If we think that all infinite points are the same, then a Poincaré's model for a hyperbolic geometry is turned to a *Klein model* for a hyperbolic geometry which uses a boundary circle and lines are straight line segment in this circle, such as those shown in Fig.6.3.2.

By a combinatorial map view, a Klein's model is nothing but a one face map geometry. This fact hints one to introduce map geometries with boundary defined in the following definition.

Definition 6.3.1 *For a map geometry* (M, μ) *without boundary and faces* $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, *if* $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ *is connected, then call* $(M, \mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ *a map geometry with boundary* $f_1, f_2, \cdots, f_l$ *and orientable or not if* (M, μ) *is orientable or not, where* $S(M)$ *denotes the locally orientable surface on which* M *is embedded.*

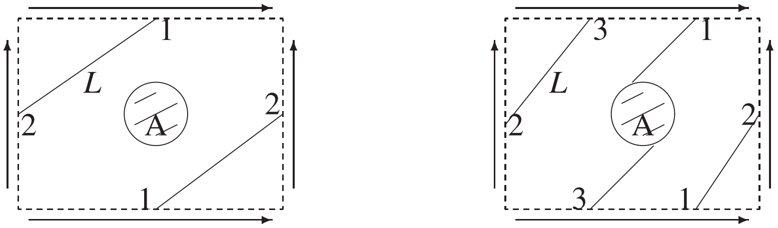


Fig.6.3.4

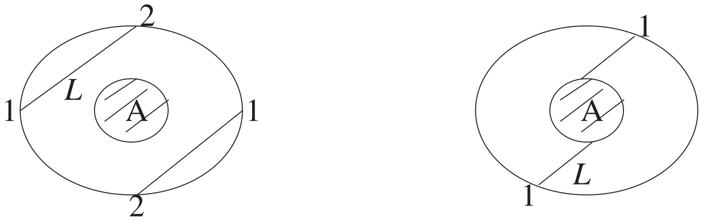


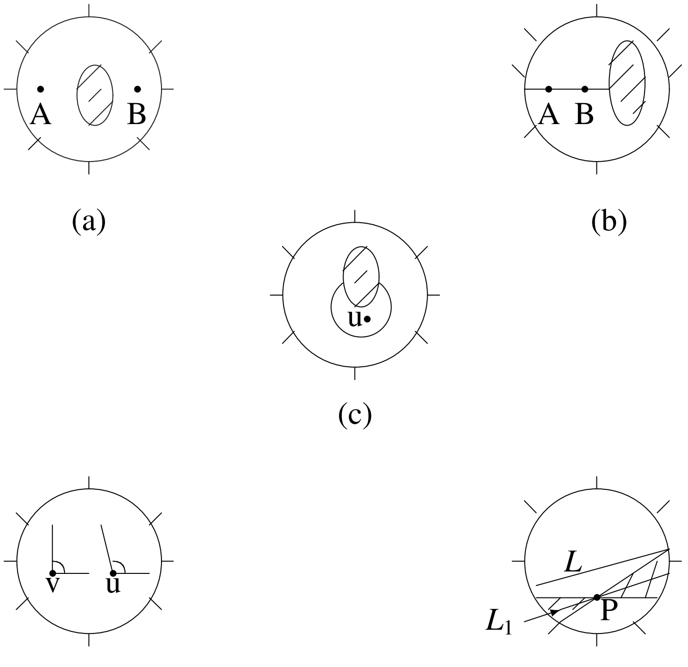
Fig.6.3.5

These s-points and s-lines in a map geometry $(M, \mu)^{-l}$ are defined as same as Definition 3.2.3 by adding an s-line terminated at the boundary of this map geometry. Two m^--lines on the torus and projective plane are shown in these Fig.6.3.4 and Fig.6.3.5, where the shade field denotes the boundary.

6.3.2 Smarandachely Map Geometry With Boundary. Indeed, there exists Smarandache geometry in map geometry with boundary convinced by results following.

Theorem 6.3.1 *For a map* M *on a locally orientable surface with order*≥ 3, *vertex valency*≥ 3 *and a face* $f \in F(M)$, *there is an angle factor* μ *such that* $(M, \mu)^{-1}$ *is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

Proof Similar to the proof of Theorem 6.2.3, we consider a map M being a planar map, an orientable map on a torus or a non-orientable map on a projective plane, respectively. We can get the assertion. In fact, by applying the property that s-lines in a map geometry with boundary are terminated at the boundary, we can get an more simpler proof for this theorem. □



(a) (b) (c) (d) (e)

Fig.6.3.6

Notice that a one face map geometry $(M,\mu)^{-1}$ with boundary is just a Klein's model for hyperbolic geometry if we choose all points being Euclidean. Similar to that of map geometry without boundary, we can also get non-geometry, anti-geometry and counter-projective geometry from that of map geometry with boundary following.

Theorem 6.3.2 *There are non-geometries in map geometries with boundary.*

Proof The proof is similar to the proof of Theorem 6.2.4 for map geometries without boundary. Each of axioms $(A_1^-)-(A_5^-)$ is hold, for example, cases $(a)-(e)$ in Fig.6.3.6, in where there are no an s-line from points A to B in (a), the line AB can not be continuously extended to indefinite in (b), the circle has gap in (c), a right angle at a Euclidean point v is not equal to a right angle at an elliptic point u in (d) and there are infinite s-lines passing through a point P not intersecting with the s-line L in (e). Whence, there are non-geometries in map geometries with boundary. □

Theorem 6.3.3 *Unless axioms $I-3$, $II-3$ $III-2$, $V-1$ and $V-2$ in the Hilbert's axiom system for a Euclid geometry, an anti-geometry can be gotten from map geometries with boundary by denial other axioms in this axiom system.*

Theorem 6.3.4 *Unless the axiom $(C3)$, a counter-projective geometry can be gotten from map geometries with boundary by denial axioms $(C1)$ and $(C2)$.*

Proof The proofs of Theorems 6.3.3 and 6.3.4 are similar to the proofs of Theorems 6.2.5 and 6.2.6. The reader can easily completes the proof. □

§6.4 CURVATURE EQUATIONS ON MAP GEOMETRY

6.4.1 Curvature on s-Line. Let (M,μ) be a map geometry with or without boundary. All points of elliptic or hyperbolic types in (M,μ) are called *non-Euclidean points*. Now let L be an s-line on (M,μ) with non-Euclisedn points $A_1, A_2, \cdots, A_n$ for an integer $n \geq 0$. Its *curvature* $R(L)$ is defined by

$$R(L) = \sum_{i=1}^{n} (\pi - \mu(A_i)).$$

An s-line L is called *Euclidean* or *non-Euclidean* if $R(L) = \pm 2\pi$ or $\neq \pm 2\pi$. Then following result characterizes s-lines on (M,μ).

Theorem 6.4.1 *An s-line without self-intersections is closed if and only if it is Euclidean.*

Proof Let L be a closed s-line without self-intersections on (M,μ) with vertices $A_1,\ A_2, \cdots, A_n$. From the Euclid geometry on plane, we know that the angle sum of an n-polygon is $(n-2)\pi$. Whence, the curvature $R(L)$ of s-line L is $\pm 2\pi$ by definition, i.e., L is Euclidean.

Now if an s-line L is Euclidean, then $R(L) = \pm 2\pi$ by definition. Thus there exist non-Euclidean points $B_1, B_2, \cdots, B_n$ such that

$$\sum_{i=1}^{n} (\pi - \mu(B_i)) = \pm 2\pi.$$

Whence, L is nothing but an n-polygon with vertices $B_1, B_2, \cdots, B_n$ on $\mathbf{R}^2$. Therefore, L is closed without self-intersection. □

Furthermore, we find conditions for an s-line to be that of regular polygon on $\mathbf{R}^2$ following.

Corollary 6.4.1 *An s-line without self-intersection passing through non-Euclidean points* $A_1, A_2, \cdots, A_n$ *is a regular polygon if and only if all points* $A_1, A_2, \cdots, A_n$ *are elliptic with*

$$\mu(A_i) = \left(1 - \frac{2}{n}\right)\pi$$

or all $A_1, A_2, \cdots, A_n$ *are hyperbolic with*

$$\mu(A_i) = \left(1 + \frac{2}{n}\right)\pi$$

for integers $1 \leq i \leq n$.

Proof If an s-line L without self-intersection passing through non-Euclidean points $A_1, A_2, \cdots, A_n$ is a regular polygon, then all points $A_1, A_2, \cdots, A_n$ must be elliptic (hyperbolic) and calculation easily shows that

$$\mu(A_i) = \left(1 - \frac{2}{n}\right)\pi \ \ or \ \ \mu(A_i) = \left(1 + \frac{2}{n}\right)\pi$$

for integers $1 \leq i \leq n$ by Theorem 9.3.5. On the other hand, if L is an s-line passing through elliptic (hyperbolic) points $A_1, A_2, \cdots, A_n$ with

$$\mu(A_i) = \left(1 - \frac{2}{n}\right)\pi \ \ or \ \ \mu(A_i) = \left(1 + \frac{2}{n}\right)\pi$$

for integers $1 \leq i \leq n$, then it is closed by Theorem 9.3.5. Clearly, L is a regular polygon with vertices $A_1, A_2, \cdots, A_n$. □

6.4.2 Curvature Equation on Map Geometry. A map $M = (\mathscr{X}_{\alpha,\beta}, \mathscr{P})$ is called *Smarandachely* if all of its vertices are elliptic (hyperbolic). Notice that these pendent vertices is not important because it can be always Euclidean or non-Euclidean. We concentrate our attention to non-separated maps. Such maps always exist circuit-decompositions. The following result characterizes Smarandachely maps.

Theorem 6.4.2 *A non-separated planar map M is Smarandachely if and only if there exist a directed circuit-decomposition*

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

of M such that one of the linear systems of equations

$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = 2\pi, \quad 1 \leq i \leq s$$

or

$$\sum_{v\in V(\overrightarrow{C}_i)} (\pi - x_v) = -2\pi, \quad 1 \leq i \leq s$$

is solvable, where $E_{\frac{1}{2}}(M)$ denotes the set of semi-arcs of M.

Proof If M is Smarandachely, then each vertex $v \in V(M)$ is non-Euclidean, i.e., $\mu(v) \neq \pi$. Whence, there exists a directed circuit-decomposition

$$E_{\frac{1}{2}}(M) = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

of semi-arcs in M such that each of them is an s-line in $(\mathbf{R}^2, \mu)$. Applying Theorem 9.3.5, we know that

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - \mu(v)) = 2\pi \text{ or } \sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - \mu(v)) = -2\pi$$

for each circuit C_i, $1 \leq i \leq s$. Thus one of the linear systems of equations

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = 2\pi, \ 1 \leq i \leq s \quad \text{or} \quad \sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = -2\pi, \ 1 \leq i \leq s$$

is solvable.

Conversely, if one of the linear systems of equations

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = 2\pi, \ 1 \leq i \leq s \quad \text{or} \quad \sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = -2\pi, \ 1 \leq i \leq s$$

is solvable, define a mapping $\mu : \mathbf{R}^2 \to [0, 4\pi)$ by

$$\mu(x) = \begin{cases} x_v & \text{if } x = v \in V(M), \\ \pi & \text{if } x \notin v(M). \end{cases}$$

Then M is a Smarandachely map on $\left(\mathbf{R}^2, \mu\right)$. This completes the proof. □

In Fig.6.4.1, we present an example of a Smarandachely planar maps with μ defined by numbers on vertices.

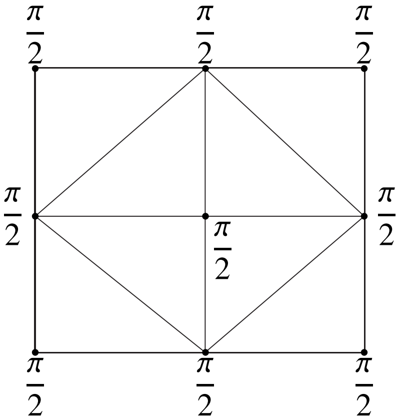


Fig.6.4.1

Let $\omega_0 \in (0,\pi)$. An s-line L is called *non-Euclidean of type* ω_0 if $R(L) = \pm 2\pi \pm \omega_0$. Similar to Theorem 6.4.1, we can get the following result.

Theorem 6.4.2 *A non-separated map M is Smarandachely if and only if there exist a directed circuit-decomposition $E_{\frac{1}{2}}(M) = \bigoplus\limits_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$ of M into s-lines of type ω_0, $\omega_0 \in (0,\pi)$ for integers $1 \le i \le s$ such that the linear systems of equations*

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = 2\pi - \omega_0, \quad 1 \le i \le s;$$

or

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = -2\pi - \omega_0, \quad 1 \le i \le s;$$

or

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = 2\pi + \omega_0, \quad 1 \le i \le s;$$

or

$$\sum_{v\in V\left(\overrightarrow{C}_i\right)} (\pi - x_v) = -2\pi + \omega_0, \quad 1 \le i \le s$$

is solvable.

§6.5 THE ENUMERATION OF MAP GEOMETRIES

6.5.1 Isomorphic Map Geometry. For classifying map geometries, the following definition on isomorphic map geometries is needed.

Definition 6.5.1 *Two map geometries* (M_1,μ_1) *and* (M_2,μ_2) *or* $(M_1,\mu_1)^{-l}$ *and* $(M_2,\mu_2)^{-l}$ *are said to be equivalent each other if there is a bijection* $\theta : M_1 \to M_2$ *such that for* $\forall u \in V(M)$, $\theta(u)$ *is euclidean, elliptic or hyperbolic if and only if* u *is euclidean, elliptic or hyperbolic.*

6.5.2 Enumerating Map Geometries. A relation for the numbers of non-equivalent map geometries with that of unrooted maps is established in the following.

Theorem 6.5.1 *Let* $\mathcal{M}$ *be a set of non-isomorphic maps of order* n *and with* m *faces. Then the number of map geometries without boundary is* $3^n|\mathcal{M}|$ *and the number of map geometries with one face being its boundary is* $3^n m|\mathcal{M}|$.

Proof By the definition of equivalent map geometries, for a given map $M \in \mathcal{M}$, there are 3^n map geometries without boundary and $3^n m$ map geometries with one face being its boundary by Theorem 6.3.1. Whence, we get $3^n|\mathcal{M}|$ map geometries without boundary and $3^n m|\mathcal{M}|$ map geometries with one face being its boundary from $\mathcal{M}$. □

We get an enumeration result for non-equivalent map geometries without boundary following.

Theorem 6.5.2 *The numbers* $n^O(\Gamma, g)$ *and* $n^N(\Gamma, g)$ *of non-equivalent orientable and non-orientable map geometries without boundary underlying a simple graph* Γ *by denial the axiom (A5) by (A5), (L5) or (R5) are*

$$n^O(\Gamma, g) = \frac{3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|},$$

and

$$n^N(\Gamma, g) = \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|},$$

where $\beta(\Gamma) = \varepsilon(\Gamma) - \nu(\Gamma) + 1$ *is the Betti number of the graph* Γ.

Proof Denote the set of non-isomorphic maps underlying the graph Γ on locally orientable surfaces by $\mathcal{M}(\Gamma)$ and the set of embeddings of the graph Γ on locally orientable surfaces by $\mathcal{E}(\Gamma)$. For a map $M, M \in \mathcal{M}(\Gamma)$, there are $\dfrac{3^{|M|}}{|\mathrm{Aut}M|}$ different map geometries without boundary by choice the angle factor μ on a vertex u such that u is Euclidean, elliptic or hyperbolic. From permutation groups, we know that

$$|\mathrm{Aut}\Gamma \times \langle\alpha\rangle| = |(\mathrm{Aut}\Gamma)_M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}| = |\mathrm{Aut}M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|.$$

Therefore, we get that

$$\begin{aligned}
n^{O}(\Gamma,g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{3^{|M|}}{|\mathrm{Aut}M|}\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}{|\mathrm{Aut}M|}\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}(\Gamma)}|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}|\mathcal{E}^{O}(\Gamma)|\\
&= \frac{3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}.
\end{aligned}$$

Similarly, we can also get that

$$\begin{aligned}
n^{N}(\Gamma,g) &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}|\mathcal{E}^{N}(\Gamma)|\\
&= \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}.
\end{aligned}$$

This completes the proof. □

By classifying map geometries with boundary, we get a result in the following.

Theorem 6.5.3 *The numbers $n^{O}(\Gamma,-g)$, $n^{N}(\Gamma,-g)$ of non-equivalent orientable, non-orientable map geometries with one face being its boundary underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are respectively*

$$n^{O}(\Gamma,-g)=\frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\left[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}\right]$$

and

$$n^{N}(\Gamma,-g)=\frac{\left(2^{\beta(\Gamma)}-1\right)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\left[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}\right],$$

where $g[\Gamma](x)$ is the genus polynomial of the graph Γ, i.e., $g[\Gamma](x)=\sum\limits_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)}g_k[\Gamma]x^k$ with $g_k[\Gamma]$ being the number of embeddings of Γ on the orientable surface of genus k.

Proof Notice that $\nu(M)-\varepsilon(M)+\phi(M)=2-2g(M)$ for an orientable map M by the Euler-Poincaré formula. Similar to the proof of Theorem 3.4.2 with the same meaning for

$\mathcal{M}(\Gamma)$, we know that

$$\begin{aligned}
n^O(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)} \frac{\phi(M)3^{|M|}}{|\mathrm{Aut}M|} \\
&= \sum_{M\in\mathcal{M}(\Gamma)} \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma)-2g(M))3^{|M|}}{|\mathrm{Aut}M|} \\
&= \sum_{M\in\mathcal{M}(\Gamma)} \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}M|} - \sum_{M\in\mathcal{M}(\Gamma)} \frac{2g(M)3^{|M|}}{|\mathrm{Aut}M|} \\
&= \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|} \sum_{M\in\mathcal{M}(\Gamma)} \frac{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}{|\mathrm{Aut}M|} \\
&\quad -\frac{2\times 3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|} \sum_{M\in\mathcal{M}(\Gamma)} \frac{g(M)|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}{|\mathrm{Aut}M|} \\
&= \frac{(\beta(\Gamma)+1)3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|} \sum_{M\in\mathcal{M}}(\Gamma)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}| \\
&\quad -\frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|} \sum_{M\in\mathcal{M}(\Gamma)} g(M)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}| \\
&= \frac{(\beta(\Gamma)+1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|} \prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|} \sum_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)} kg_k[\Gamma] \\
&= \frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\left[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}\Big|_{x=1}\right].
\end{aligned}$$

by Theorem 6.5.1.

Notice that $n^L(\Gamma,-g)=n^O(\Gamma,-g)+n^N(\Gamma,-g)$ and the number of re-embeddings an orientable map M on surfaces is $2^{\beta(M)}$ (See also [Mao10] or [Mao34] for details). We know that

$$\begin{aligned}
n^L(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)} \frac{2^{\beta(M)}\times 3^{|M|}\phi(M)}{|\mathrm{Aut}M|} \\
&= 2^{\beta(M)}n^O(\Gamma,-g).
\end{aligned}$$

Whence, we get that

$$\begin{aligned}
n^N(\Gamma,-g) &= (2^{\beta(M)}-1)n^O(\Gamma,-g) \\
&= \frac{\left(2^{\beta(M)}-1\right)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\left[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}\Big|_{x=1}\right].
\end{aligned}$$

This completes the proof. □

§6.6 RESEARCH PROBLEMS

6.6.1 A complete Hilbert axiom system for a Euclid geometry contains axioms $I-i, 1 \leq i \leq 8$; $II-j, 1 \leq j \leq 4$; $III-k, 1 \leq k \leq 5$; $IV-1$ and $V-l, 1 \leq l \leq 2$, which can be also applied to the geometry of space. Unless $I-i, 4 \leq i \leq 8$, other axioms are presented in Section 6.2. Each of axioms $I-i, 4 \leq i \leq 8$ is described in the following.

$I-4$ *For three non-collinear points A, B and C, there is one and only one plane passing through them.*

$I-5$ *Each plane has at least one point.*

$I-6$ *If two points A and B of a line L are in a plane $\sum$, then every point of L is in the plane $\sum$.*

$I-7$ *If two planes $\sum_1$ and $\sum_2$ have a common point A, then they have another common point B.*

$I-8$ *There are at least four points not in one plane.*

By the Hilbert's axiom system, the following result for parallel planes can be obtained.

(**T**) *Passing through a given point A exterior to a given plane $\sum$ there is one and only one plane parallel to $\sum$.*

This result seems like the Euclid's fifth axiom. Similar to the Smarandache's notion, we present problems by denial this result for geometry of space following.

Problem 6.6.1 *Construct a geometry of space by denial the parallel theorem of planes with*

(T_1^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that no parallel plane to $\sum$ passing through the point A.*

(T_2^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that there are finite parallel planes to $\sum$ passing through the point A.*

(T_3^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that there are infinite parallel planes to $\sum$ passing through the point A.*

Problem 6.6.2 *Similar to that of Iseri's idea, define points of elliptic, Euclidean, or hyperbolic type in $\mathbf{R}^3$ and apply these Plato polyhedrons to construct Smarandache geometry of space $\mathbf{R}^3$.*

Problem 6.6.3 *Similar to that of map geometry and apply graphs in* $\mathbf{R}^3$ *to construct Smarandache geometry of space* $\mathbf{R}^3$.

Problem 6.6.4 *For an integer* $n, n \geq 4$, *define Smarandache geometry in* $\mathbf{R}^n$ *by denial some axioms for an Euclid geometry in* $\mathbf{R}^n$ *and construct them.*

6.6.2 The terminology of *map geometry* was first appeared in [Mao9], which enables one to find non-homogenous spaces from already known homogenous spaces and is also a typical example for application combinatorial maps to metric geometries. Among them there are many problems not solved yet until today. Here we would like to describe some of them.

Problem 6.6.5 *For a given graph* G, *determine non-equivalent map geometries underlying a graph* G, *particularly, underlying graphs* K_n *or* $K(m,n),\ m,n \geq 4$ *and enumerate them.*

Problem 6.6.6 *For a given locally orientable surface* S, *determine non-equivalent map geometries on* S, *such as a sphere, a torus or a projective plane,* $\cdots$ *and enumerate them.*

Problem 6.6.7 *Find characteristics for equivalent map geometries or establish new ways for classifying map geometries.*

Problem 6.6.8 *Whether can we rebuilt an intrinsic geometry on surfaces, such as a sphere, a torus or a projective plane,* $\cdots$, *etc. by map geometry?*

CHAPTER 7.

Planar Map Geometry

As we seen, a map geometry (M, μ) is nothing but a map M associate vertices with an angle factor μ. This means that there are finite non-Euclidean points in map geometry (M, μ). However, a map is a graph on surface, i.e., a geometrical graph. We can also generalize the angle factor to edges, i.e., associate points in edges of map with an angle function and then find the behavior of points, straight lines, polygons and circles, $\cdots$, i.e., fundamental elements in Euclid geometry on plane. In this case, the situation is more complex since a point maybe an elliptic, Euclidean or hyperbolic and a polygon maybe an s-line, $\cdots$, etc.. We introduce such map geometry on plane, discuss points with a classification of edges in Section 7.1, lines with curvature in Section 7.2. The polygons, including the number of sides, internal angle sum, area and circles on planar map geometry are discussed in Sections 7.3 and 7.4. For finding the behavior of s-lines, we introduce line bundles, motivated by the Euclid' s fifth postulate and determine their behavior on planar map geometry in Section 7.5. All of these materials will be used for establishing relations of an integral curve with a differential equation system in a pseudo-plane geometry and continuous phenomena with that of discrete phenomena in following chapters.

§7.1 POINTS IN PLANAR MAP GEOMETRY

7.1.1 Angle Function on Edge. The points in a map geometry are classified into three classes: *elliptic, Euclidean* and *hyperbolic*. There are only finite non-Euclidean points considered in Chapter 6 because we had only defined an elliptic, Euclidean or a hyperbolic point on vertices of map M. In planar map geometry, we can present an even more delicate consideration for Euclidean or non-Euclidean points and find infinite non-Euclidean points in a plane.

Let (M,μ) be a planar map geometry on plane $\sum$. Choose vertices $u,v\in V(M)$. A mapping is called an *angle function between u and v* if there is a smooth monotone mapping $f:\ \sum\to\sum$ such that $f(u)=\frac{\rho_M(u)\mu(u)}{2}$ and $f(v)=\frac{\rho_M(v)\mu(v)}{2}$. Not loss of generality, we can assume that there is an angle function on each edge in a planar map geometry. Then we know a result following.

Theorem 7.1.1 *A planar map geometry (M,μ) has infinite non-Euclidean points if and only if there is an edge $e=(u,v)\in E(M)$ such that $\rho_M(u)\mu(u)\neq\rho_M(v)\mu(v)$, or $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u\in V(M)$, or a loop $(u,u)\in E(M)$ attaching a non-Euclidean point u.*

Proof If there is an edge $e=(u,v)\in E(M)$ such that $\rho_M(u)\mu(u)\neq\rho_M(v)\mu(v)$, then at least one of vertices u and v in (M,μ) is non-Euclidean. Not loss of generality, we assume the vertex u is non-Euclidean.

If u and v are elliptic or u is elliptic but v is Euclidean, then by the definition of angle functions, the edge (u,v) is correspondent with an angle function $f:\sum\to\sum$ such that $f(u)=\frac{\rho_M(u)\mu(u)}{2}$ and $f(v)=\frac{\rho_M(v)\mu(v)}{2}$, each points is non-Euclidean in $(u,v)\setminus\{v\}$. If u is elliptic but v is hyperbolic, i.e., $\rho_M(u)\mu(u)<2\pi$ and $\rho_M(v)\mu(v)>2\pi$, since f is smooth and monotone on (u,v), there is one and only one point x^* in (u,v) such that $f(x^*)=\pi$. Thereby there are infinite non-Euclidean points on (u,v).

Similar discussion can be gotten for the cases that u and v are both hyperbolic, or u is hyperbolic but v is Euclidean, or u is hyperbolic but v is elliptic.

If $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u\in V(M)$, then each point on an edges is a non-Euclidean point. Consequently, there are infinite non-Euclidean points in (M,μ).

Now if there is a loop $(u,u)\in E(M)$ and u is non-Euclidean, then by definition, each point v on the loop (u,u) satisfying that $f(v)>$ or $<\pi$ according to $\rho_M(u)\mu(u)>\pi$ or $<\pi$. Therefore there are also infinite non-Euclidean points on the loop (u,u).

On the other hand, if there are no an edge $e = (u, v) \in E(M)$ such that $\rho_M(u)\mu(u) \neq \rho_M(v)\mu(v)$, i.e., $\rho_M(u)\mu(u) = \rho_M(v)\mu(v)$ for $\forall (u, v) \in E(M)$, or there are no vertices $u \in V(M)$ such that $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall$, or there are no loops $(u, u) \in E(M)$ with a non-Euclidean point u, then all angle functions on these edges of M are an constant π. Therefore there are no non-Euclidean points in the map geometry (M, μ). This completes the proof. □

Characterizing Euclidean points in planar map geometry (M, μ), we get the following result.

Theorem 7.1.2 *Let (M, μ) be a planar map geometry on plane $\sum$. Then*

(1) *Every point in $\sum \setminus E(M)$ is a Euclidean point;*

(2) *There are infinite Euclidean points on M if and only if there exists an edge $(u, v) \in E(M)$ ($u = v$ or $u \neq v$) such that u and v are both Euclidean.*

Proof By the definition of angle functions, we know that every point is Euclidean if it is not on M. So the assertion (1) is true.

According to the proof of Theorem 7.1.1, there are only finite Euclidean points unless there is an edge $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$. In this case, there are infinite Euclidean points on the edge (u, v). Thereby the assertion (2) is also holds. □

7.1.2 Edge Classification. According to Theorems 7.1.1 and 7.1.2, we classify edges in a planar map geometry (M, μ) into six classes.

C_E^1 **(Euclidean-elliptic edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ but $\rho_M(v)\mu(v) < 2\pi$.*

C_E^2 **(Euclidean-Euclidean edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ and $\rho_M(v)\mu(v) = 2\pi$.*

C_E^3 **(Euclidean-hyperbolic edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ but $\rho_M(v)\mu(v) > 2\pi$.*

C_E^4 **(elliptic-elliptic edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) < 2\pi$ and $\rho_M(v)\mu(v) < 2\pi$.*

C_E^5 **(elliptic-hyperbolic edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) < 2\pi$ but $\rho_M(v)\mu(v) > 2\pi$.*

C_E^6 **(hyperbolic-hyperbolic edges)**: *edges $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) > 2\pi$ and $\rho_M(v)\mu(v) > 2\pi$.*

In Fig.7.1.1$(a)-(f)$, these s-lines passing through an edge in one of classes of C_E^1-C_E^6 are shown, where u is elliptic and v is Euclidean in (a), u and v are both Euclidean in (b), u is Euclidean but v is hyperbolic in (c), u and v are both elliptic in (d), u is elliptic but v is hyperbolic in (e) and u and v are both hyperbolic in (f), respectively.

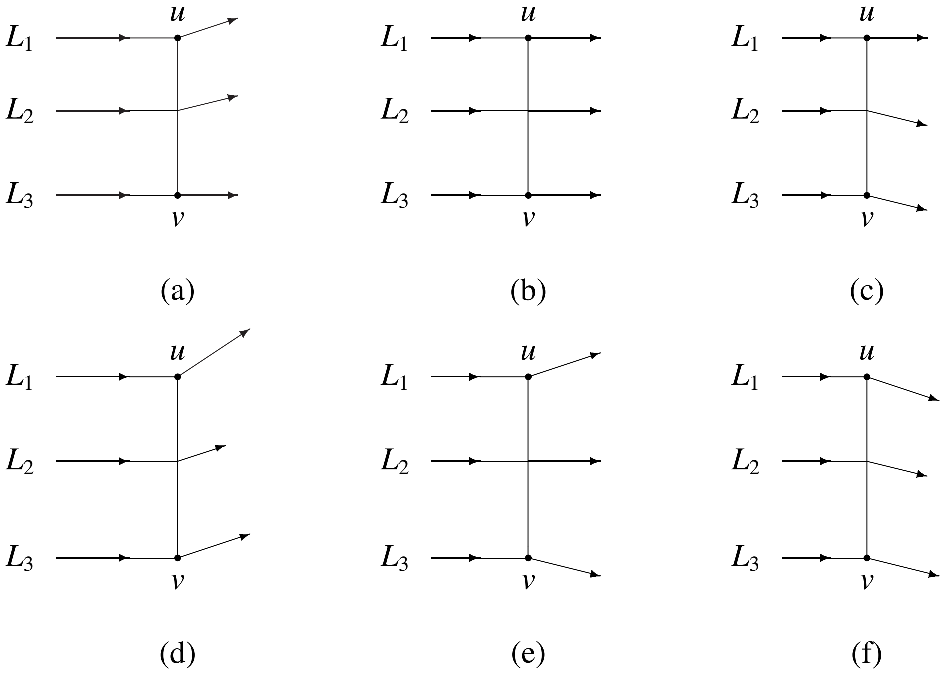


Fig.7.1.1

Denote by $V_{el}(M), V_{eu}(M)$ and $V_{hy}(M)$ the respective sets of elliptic, Euclidean and hyperbolic points in $V(M)$ in a planar map geometry (M, μ). Then we get a result as in the following.

Theorem 7.1.3 *Let (M, μ) be a planar map geometry. Then*

$$\sum_{u \in V_{el}(M)} \rho_M(u) + \sum_{v \in V_{eu}(M)} \rho_M(v) + \sum_{w \in V_{hy}(M)} \rho_M(w) = 2 \sum_{i=1}^{6} |C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6} |C_E^i| + 2.$$

where $\phi(M)$ denotes the number of faces of a map M.

Proof Notice that

$$|V(M)| = |V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| \text{ and } |E(M)| = \sum_{i=1}^{6} |C_E^i|$$

for a planar map geometry (M, μ). By two well-known results

$$\sum_{v \in V(M)} \rho_M(v) = 2|E(M)| \text{ and } |V(M)| - |E(M)| + \phi(M) = 2$$

for a planar map M, we know that

$$\sum_{u \in V_{el}(M)} \rho_M(u) + \sum_{v \in V_{eu}(M)} \rho_M(v) + \sum_{w \in V_{hy}(M)} \rho_M(w) = 2 \sum_{i=1}^{6} |C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6} |C_E^i| + 2. \qquad \square$$

§7.2 LINES IN PLANAR MAP GEOMETRY

The situation of s-lines in a planar map geometry (M, μ) is more complex. Here an s-line maybe open or closed, with or without self-intersections in a plane. We discuss all of these s-lines and their behaviors in this section, .

7.2.1 Lines in Planar Map Geometry. As we have seen in Chapter 6, s-lines in a planar map geometry (M, μ) can be classified into three classes.

C_L^1 **(opened lines without self-intersections)**: *s-lines in* (M, μ) *have an infinite number of continuous s-points without self-intersections and endpoints and may be extended indefinitely in both directions.*

C_L^2 **(opened lines with self-intersections)**: *s-lines in* (M, μ) *have an infinite number of continuous s-points and self-intersections but without endpoints and may be extended indefinitely in both directions.*

C_L^3**(closed lines)**: *s-lines in* (M, μ) *have an infinite number of continuous s-points and will come back to the initial point as we travel along any one of these s-lines starting at an initial point.*

By this classification, a straight line in a Euclid plane is nothing but an opened s-line without non-Euclidean points. Certainly, s-lines in a planar map geometry (M,μ) maybe contain non-Euclidean points. In Fig.7.2.1, these s-lines shown in $(a),(b)$ and (c) are opened s-line without self-intersections, opened s-line with a self-intersection and closed s-line with A,B,C,D and E non-Euclidean points, respectively.

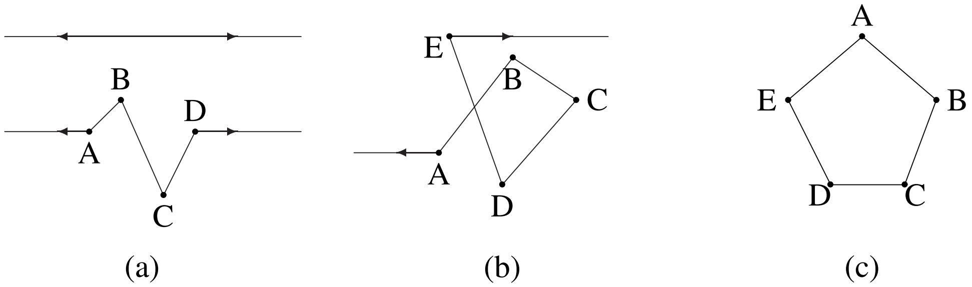


Fig.7.2.1

Notice that a closed s-line in a planar map geometry maybe also has self-intersections. A closed s-line is said to be *simply closed* if it has no self-intersections, such as the s-line in Fig.7.2.1(c). For simply closed s-lines, we know the following result.

Theorem 7.2.1 *Let (M,μ) be a planar map geometry. An s-line L in (M,μ) passing through n non-Euclidean points $x_1,x_2,\cdots,x_n$ is simply closed if and only if*

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an s-point x_i, $1 \le i \le n$.

Proof By results in Euclid geometry of plane, we know that the angle sum of an n-polygon is $(n-2)\pi$. In a planar map geometry (M,μ), a simply closed s-line L passing through n non-Euclidean points $x_1,x_2,\cdots,x_n$ is nothing but an n-polygon with vertices $x_1,x_2,\cdots,x_n$. Whence, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi.$$

Now if a simply s-line L passing through n non-Euclidean points $x_1,x_2,\cdots,x_n$ with

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi$$

held, then L is nothing but an n-polygon with vertices $x_1, x_2, \cdots, x_n$. Therefore, L is simply closed. □

By applying Theorem 7.2.1, we can also find conditions for an opened s-line with or without self-intersections.

Theorem 7.2.2 *Let (M,μ) be a planar map geometry. An s-line L in (M,μ) passing through n non-Euclidean points $x_1, x_2, \cdots, x_n$ is opened without self-intersections if and only if s-line segments $x_ix_{i+1}, 1 \leq i \leq n-1$ are not intersect two by two and*

$$\sum_{i=1}^{n} f(x_i) \geq (n-1)\pi.$$

Proof By the Euclid's fifth postulate for a plane geometry, two straight lines will meet on the side on which the angles less than two right angles if we extend them to indefinitely. Now for an s-line L in a planar map geometry (M,μ), if it is opened without self-intersections, then for any integer $i, 1 \leq i \leq n-1$, s-line segments x_ix_{i+1} will not intersect two by two and the s-line L will also not intersect before it enters x_1 or leaves x_n.

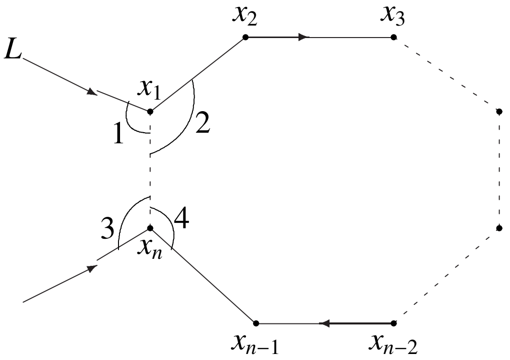


Fig.7.2.2

Now look at Fig.7.2.2, in where line segment x_1x_n is an added auxiliary s-line segment. We know that

$$\angle 1 + \angle 2 = f(x_1) \text{ and } \angle 3 + \angle 4 = f(x_n).$$

According to Theorem 7.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{i=2}^{n-1} f(x_i) = (n-2)\pi$$

and

$$\angle 1 + \angle 3 \geq \pi$$

Therefore, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi + \angle 1 + \angle 3 \geq (n-1)\pi. \qquad \square$$

For opened s-lines with self-intersections, we know a result in the following.

Theorem 7.2.3 Let (M,μ) be a planar map geometry. An s-line L in (M,μ) passing through n non-Euclidean points $x_1, x_2, \cdots, x_n$ is opened only with l self-intersections if and only if there exist integers i_j and s_{i_j}, $1 \leq j \leq l$ with $1 \leq i_j, s_{i,j} \leq n$ and $i_j \neq i_t$ if $t \neq j$ such that

$$(s_{i_j} - 2)\pi \ < \ \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) \ < \ (s_{i_j} - 1)\pi.$$

Proof If an s-line L passing through s-points $x_{t+1}, x_{t+2}, \cdots, x_{t+s_t}$ only has one self-intersection point, let us look at Fig.7.2.3 in where $x_{t+1}x_{t+s_t}$ is an added auxiliary s-line segment.

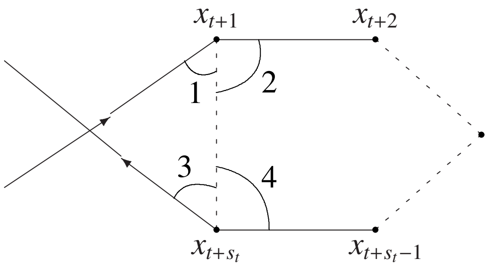


Fig.7.2.3

We know that

$$\angle 1 + \angle 2 = f(x_{t+1}) \text{ and } \angle 3 + \angle 4 = f(x_{t+s_t}).$$

Similar to the proof of Theorem 7.2.2, by Theorem 7.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{j=2}^{s_t-1} f(x_{t+j}) = (s_t - 2)\pi$$

and

$$\angle 1 + \angle 3 \ < \ \pi.$$

Whence, we get that

$$(s_t - 2)\pi < \sum_{j=1}^{s_t} f(x_{t+j}) < (s_t - 1)\pi.$$

Therefore, if L is opened only with l self-intersection points, we can find integers i_j and s_{i_j}, $1 \le j \le l$ with $1 \le i_j, s_{i,j} \le n$ and $i_j \ne i_t$ if $t \ne j$ such that L passing through $x_{i_j+1}, x_{i_j+2}, \cdots, x_{i_j+s_j}$ only has one self-intersection point. By the previous discussion, we know that

$$(s_{i_j} - 2)\pi < \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) < (s_{i_j} - 1)\pi.$$

This completes the proof. □

7.2.2 Curve Curvature. Notice that all s-lines considered in this section are consisted of line segments or rays in Euclid plane geometry. If the length of each line segment tends to zero, then we get a curve at the limitation in the usually sense. Whence, an s-line in a planar map geometry can be also seen as a discretion of plane curve.

Generally, the curvature at a point of a curve C is a measure of how quickly the tangent vector changes direction with respect to the length of arc, such as those of the Gauss curvature, the Riemann curvature, $\cdots$, etc.. In Fig.7.2.4 we present a smooth curve and the changing of tangent vectors.

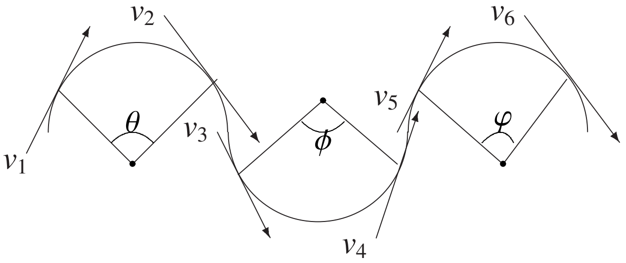


Fig.7.2.4

To measure the changing of vector v_1 to v_2, a simpler way is by the changing of the angle between vectors v_1 and v_2. If a curve $C = f(s)$ is smooth, then the changing rate of the angle between two tangent vector with respect to the length of arc, i.e., $\frac{df}{ds}$ is continuous. For example, as we known in the differential geometry, the Gauss curvature at every point of a circle $x^2 + y^2 = r^2$ of radius r is $\frac{1}{r}$. Whence, the changing of the angle from vectors v_1 to v_2 is

$$\int\limits_A^B \frac{1}{r}ds = \frac{1}{r}|\widehat{AB}| = \frac{1}{r}r\theta = \theta.$$

By results in Euclid plane geometry, we know that θ is also the angle between vectors v_1 and v_2. As we illustrated in Subsection 7.2.1, an s-line in a planar map geometry is consisted by line segments or rays. Therefore, the changing rate of the angle between two tangent vector with respect to the length of arc is not continuous. Similar to the definition of the set curvature in the reference [AlZ1], we present a discrete definition for the curvature of s-lines in this case following.

Definition 7.2.1 *Let L be an s-line in a planar map geometry (M,μ) with the set W of non-Euclidean points. The curvature $\omega(L)$ of L is defined by*

$$\omega(L) = \int\limits_W (\pi - \varpi(p)),$$

where $\varpi(p) = f(p)$ if p is on an edge (u,v) in map M embedded on plane $\sum$ with an angle function $f : \sum \to \sum$.

In differential geometry, the *Gauss mapping* and the *Gauss curvature* on surfaces are defined as follows:

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. The mapping $N : \mathcal{S} \to S^2$ takes its value in the unit sphere

$$S^2 = \{(x,y,z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. The map $N : \mathcal{S} \to S^2$, thus defined, is called a Gauss mapping and the determinant of $K(p) = d\mathbf{N}_p$ a Gauss curvature.

We know that for a point $p \in \mathcal{S}$ such that the Gaussian curvature $K(p) \neq 0$ and a connected neighborhood V of p with K does not change sign,

$$K(p) = \lim_{A\to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$.

The well-known *Gauss-Bonnet theorem* for a compact surface says that

$$\int\int_S K d\sigma = 2\pi\chi(S),$$

for any orientable compact surface S.

For a simply closed s-line, we also have a result similar to the Gauss-Bonnet theorem, which can be also seen as a discrete Gauss-Bonnet theorem on a plane.

Theorem 7.2.4 *Let* L *be a simply closed* s*-line passing through* n *non-Euclidean points* $x_1, x_2, \cdots, x_n$ *in a planar map geometry* (M, μ)*. Then* $\omega(L) = 2\pi$.

Proof According to Theorem 7.2.1, we know that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an s-point $x_i, 1 \leq i \leq n$. Whence, by Definition 7.2.1 we know that

$$\begin{aligned} \omega(L) &= \int\limits_{\{x_i; 1\leq i\leq n\}} (\pi - f(x_i)) = \sum_{i=1}^{n}(\pi - f(x_i)) \\ &= \pi n - \sum_{i=1}^{n} f(x_i) = \pi n - (n-2)\pi = 2\pi. \end{aligned}$$ □

Similarly, we also get the sum of curvatures on the planar map M in (M, μ) following.

Theorem 7.2.5 *Let* (M, μ) *be a planar map geometry. Then the sum* $\omega(M)$ *of curvatures on edges in a map* M *is* $\omega(M) = 2\pi s(M)$*, where* $s(M)$ *denotes the sum of length of edges in* M.

Proof Notice that the sum $\omega(u, v)$ of curvatures on an edge (u, v) of M is

$$\omega(u,v) = \int\limits_{v}^{u} (\pi - f(s))ds = \pi|\widehat{(u,v)}| - \int\limits_{v}^{u} f(s)ds.$$

Since M is a planar map, each of its edges appears just two times with an opposite direction. Whence, we get that

$$\begin{aligned} \omega(M) &= \sum_{(u,v)\in E(M)} \omega(u,v) + \sum_{(v,u)\in E(M)} \omega(v,u) \\ &= \pi \sum_{(u,v)\in E(M)} \left(|\widehat{(u,v)}| + |\widehat{(v,u)}|\right) - \left(\int\limits_{v}^{u} f(s)ds + \int\limits_{u}^{v} f(s)ds \right) = 2\pi s(M) \end{aligned}$$ □

Notice that if$s(M) = 1$, Theorem 7.2.5 turns to the Gauss-Bonnet theorem for sphere.

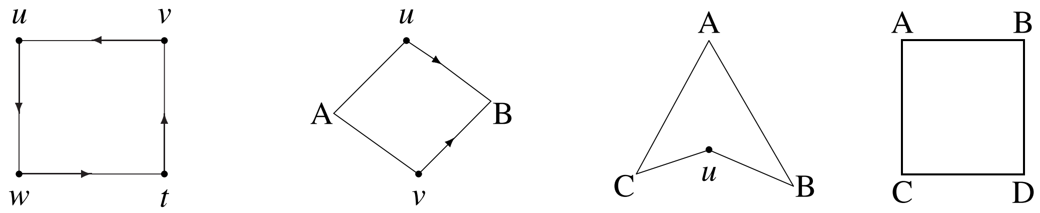
§7.3 POLYGONS IN PLANAR MAP GEOMETRY

7.3.1 Polygon in Planar Map Geometry. In the Euclid plane geometry, we have encountered triangles, quadrilaterals, $\cdots$, and generally, n-polygons, i.e., these graphs on a plane with n straight line segments not on the same line connected with one after another. There are no 1 and 2-polygons in a Euclid plane geometry since every point is Euclidean. The definition of n-polygons in planar map geometry (M,μ) is similar to that of Euclid plane geometry.

Definition 7.3.1 *An n-polygon in a planar map geometry* (M,μ) *is defined to be a graph in* (M,μ) *with n s-line segments two by two without self-intersections and connected with one after another.*

Although their definition is similar, the situation is more complex in a planar map geometry (M,μ). We have found a necessary and sufficient condition for 1-polygon in Theorem 7.2.1, i.e., 1-polygons maybe exist in a planar map geometry. In general, we can find n-polygons in a planar map geometry for any integer $n, n \geq 1$.

Examples of polygon in a planar map geometry (M,μ) are shown in Fig.7.3.1, in where (a) is a 1-polygon with u, v, w and t being non-Euclidean points, (b) is a 2-polygon with vertices A, B and non-Euclidean points u, v, (c) is a triangle with vertices A, B, C and a non-Euclidean point u and (d) is a quadrilateral with vertices A, B, C and D.



(a) (b) (c) (d)

Fig.7.3.1

Theorem 7.3.1 *There exists a* 1*-polygon in a planar map geometry* (M,μ) *if and only if there are non-Euclidean points* $u_1, u_2, \cdots, u_l$ *with* $l \geq 3$ *such that*

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

where $f(u_i)$ *denotes the angle function value at the point* u_i, $1 \leq i \leq l$.

Proof According to Theorem 7.2.1, an s-line passing through l non-Euclidean points $u_1, u_2, \cdots, u_l$ is simply closed if and only if

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

i.e., 1-polygon exists in (M,μ) if and only if there are non-Euclidean points $u_1, u_2, \cdots, u_l$ with the above formula hold.

Whence, we only need to prove $l \geq 3$. Since there are no 1-polygons or 2-polygons in a Euclid plane geometry, we must have $l \geq 3$ by the Hilbert's axiom $I-2$. In fact, for $l=3$ we can really find a planar map geometry (M,μ) with a 1-polygon passing through three non-Euclidean points u, v and w. Look at Fig.7.3.2,

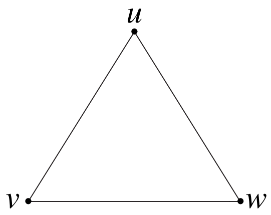


Fig.7.3.2

in where the angle function values are $f(u) = f(v) = f(w) = \frac{2}{3}\pi$ at u, v and w. □

Similarly, for 2-polygons we know the following result.

Theorem 7.3.2 *There are* 2*-polygons in a planar map geometry* (M,μ) *only if there are at least one non-Euclidean point in* (M,μ)*.*

Proof In fact, if there is a non-Euclidean point u in (M,μ), then each straight line enter u will turn an angle $\theta = \pi - \frac{f(u)}{2}$ or $\frac{f(u)}{2} - \pi$ from the initial straight line dependent on that u is elliptic or hyperbolic. Therefore, we can get a 2-polygon in (M,μ) by choice a straight line AB passing through Euclidean points in (M,μ), such as the graph shown in Fig.7.3.3.

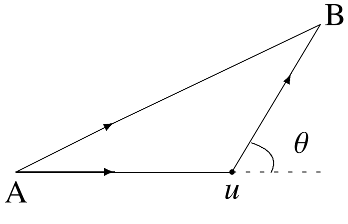


Fig.7.3.3

This completes the proof. □

For the existence of n-polygons with $n \geq 3$, we have a general result as in the following.

Theorem 7.3.3 *For any integer $n, n \geq 3$, there are n-polygons in a planar map geometry (M, μ).*

Proof Since in Euclid plane geometry, there are n-polygons for any integer $n, n \geq 3$. Therefore, there are also n-polygons in a planar map geometry (M, μ) for any integer $n, n \geq 3$. □

7.3.2 Internal Angle Sum. For the sum of the internal angles in an n-polygon, we have the following result.

Theorem 7.3.4 *Let $\prod$ be an n-polygon in a map geometry with its edges passing through non-Euclidean points $x_1, x_2, \cdots, x_l$. Then the sum of internal angles in $\prod$ is*

$$(n + l - 2)\pi - \sum_{i=1}^{l} f(x_i),$$

where $f(x_i)$ denotes the value of the angle function f at the point x_i, $1 \leq i \leq l$.

Proof Denote by U, V the sets of elliptic points and hyperbolic points in $x_1, x_2, \cdots, x_l$ and $|U| = p, |V| = q$, respectively. If an s-line segment passes through an elliptic point u, add an auxiliary line segment AB in the plane as shown in Fig.7.3.4(1).

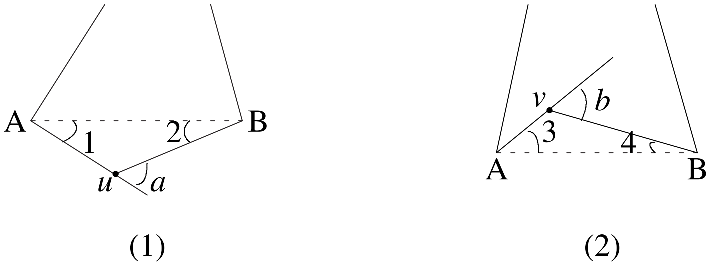


Fig.7.3.4

Then we get that

$$\angle \mathrm{a} = \angle 1 + \angle 2 = \pi - f(u).$$

If an s-line passes through a hyperbolic point v, also add an auxiliary line segment

AB in the plane as that shown in Fig.7.3.4(2). Then we get that

$$\text{angle } b = \text{angle3} + \text{angle4} = f(v) - \pi.$$

Since the sum of internal angles of an n-polygon in a plane is $(n-2)\pi$ whenever it is a convex or concave polygon, we know that the sum of the internal angles in $\prod$ is

$$\begin{aligned}&(n-2)\pi + \sum_{x\in U}(\pi - f(x)) - \sum_{y\in V}(f(y) - \pi)\\ &= (n+p+q-2)\pi - \sum_{i=1}^{l} f(x_i)\\ &= (n+l-2)\pi - \sum_{i=1}^{l} f(x_i).\end{aligned}$$

This completes the proof. □

A triangle is called *Euclidean, elliptic* or *hyperbolic* if its edges only pass through one kind of Euclidean, elliptic or hyperbolic points. As a consequence of Theorem 7.3.4, we get the sum of the internal angles of a triangle in a map geometry which is consistent with these already known results.

Corollary 7.3.1 *Let $\triangle$ be a triangle in a planar map geometry (M,μ). Then*

(1) *the sum of its internal angles is equal to π if $\triangle$ is Euclidean;*
(2) *the sum of its internal angles is less than π if $\triangle$ is elliptic;*
(3) *the sum of its internal angles is more than π if $\triangle$ is hyperbolic.*

Proof Notice that the sum of internal angles of a triangle is

$$\pi + \sum_{i=1}^{l}(\pi - f(x_i))$$

if it passes through non-Euclidean points $x_1, x_2, \cdots, x_l$. By definition, if these x_i, $1 \le i \le l$ are one kind of Euclidean, elliptic, or hyperbolic, then we have that $f(x_i) = \pi$, or $f(x_i) < \pi$, or $f(x_i) > \pi$ for any integer i, $1 \le i \le l$. Whence, the sum of internal angles of a Euclidean, elliptic or hyperbolic triangle is equal to, or lees than, or more than π. □

7.3.3 Polygon Area. As it is well-known, calculation for the area $A(\triangle)$ of a triangle $\triangle$ with two sides a, b and the value of their include angle θ or three sides a, b and c in a Euclid plane is simple. Formulae for its area are

$$A(\triangle) = \frac{1}{2}ab\sin\theta \text{ or } A(\triangle) = \sqrt{s(s-a)(s-b)(s-c)},$$

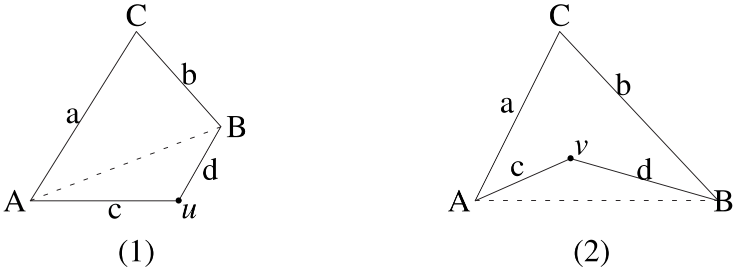
where $s = \frac{1}{2}(a+b+c)$. But in a planar map geometry, calculation for the area of a triangle is complex since each of its edge maybe contains non-Euclidean points. Where, we only present a programming for calculation the area of a triangle in a planar map geometry.

STEP 1. *Divide a triangle into triangles in a Euclid plane such that no edges contain non-Euclidean points unless their endpoints;*

STEP 2. *Calculate the area of each triangle;*

STEP 3. *Sum up all of areas of these triangles to get the area of the given triangle in a planar map geometry.*

The simplest cases for triangle is the cases with only one non-Euclidean point such as those shown in Fig.7.3.5(1) and (2) with an elliptic point u or with a hyperbolic point v.



(1) (2)

Fig.7.3.5

Add an auxiliary line segment AB in Fig.7.3.5. Then by formulae in the plane trigonometry, we know that

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} + \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

for case (1) and

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} - \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

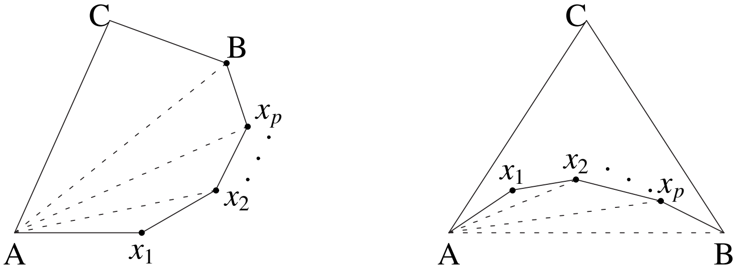
for case (2) in Fig.7.3.5, where

$$t = \sqrt{c^2 + d^2 - 2cd\cos\frac{f(x)}{2}}$$

with $x = u$ or v and

$$s_1 = \frac{1}{2}(a+b+t), \quad s_2 = \frac{1}{2}(c+d+t).$$

Generally, let $\triangle ABC$ be a triangle with its edge AB passing through p elliptic or p hyperbolic points $x_1, x_2, \cdots, x_p$ simultaneously, as those shown in Fig.7.3.6(1) and (2).



(1) (2)

Fig.7.3.6

Where $|AC| = a$, $|BC| = b$ and $|Ax_1| = c_1$, $|x_1x_2| = c_2, \cdots, |x_{p-1}x_p| = c_p$ and $|x_pB| = c_{p+1}$. Adding auxiliary line segments $Ax_2, Ax_3, \cdots, Ax_p, AB$ in Fig.7.3.6, then we can find its area by the programming STEP 1 to STEP 3. By formulae in the plane trigonometry, we get that

$$|Ax_2| = \sqrt{c_1^2 + c_2^2 - 2c_1c_2\cos\frac{f(x_1)}{2}},$$

$$\angle Ax_2x_1 = \cos^{-1}\frac{c_1^2 - c_2^2 - |Ax_1|^2}{2c_2|Ax_2|},$$

$$\angle Ax_2x_3 = \frac{f(x_2)}{2} - \angle Ax_2x_1 \text{ or } 2\pi - \frac{f(x_2)}{2} - \angle Ax_2x_1,$$

$$|Ax_3| = \sqrt{|Ax_2|^2 + c_3^2 - 2|Ax_2|c_3\cos(\frac{f(x_2)}{2} - \angle Ax_2x_3)},$$

$$\angle Ax_3x_2 = \cos^{-1}\frac{|Ax_2|^2 - c_3^2 - |Ax_3|^2}{2c_3|Ax_3|},$$

$$\angle Ax_2x_3 = \frac{f(x_3)}{2} - \angle Ax_3x_2 \text{ or } 2\pi - \frac{f(x_3)}{2} - \angle Ax_3x_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and generally, we get that

$$|AB| = \sqrt{|Ax_p|^2 + c_{p+1}^2 - 2|Ax_p|c_{p+1}\cos\angle Ax_pB}.$$

Then the area of the triangle $\triangle ABC$ is

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p - a)(s_p - b)(s_p - |AB|)} \\ &+ \sum_{i=1}^{p} \sqrt{s_i(s_i - |Ax_i|)(s_i - c_{i+1})(s_i - |Ax_{i+1}|)} \end{aligned}$$

for case (1) and

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p-a)(s_p-b)(s_p-|AB|)} \\ &- \sum_{i=1}^{p} \sqrt{s_i(s_i-|Ax_i|)(s_i-c_{i+1})(s_i-|Ax_{i+1}|)} \end{aligned}$$

for case (2) in Fig.7.3.6, where

$$s_i = \frac{1}{2}(|Ax_i| + c_{i+1} + |Ax_{i+1}|)$$

for any integer $i,\ 1 \leq i \leq p-1$ and

$$s_p = \frac{1}{2}(a + b + |AB|).$$

Certainly, this programming can be also applied to calculate the area of an n-polygon in planar map geometry in general.

§7.4 CIRCLES IN PLANAR MAP GEOMETRY

The length of an s-line segment in planar map geometry is defined in the following.

Definition 7.4.1 *The length $|AB|$ of an s-line segment AB consisted by k straight line segments $AC_1, C_1C_2, C_2C_3, \cdots, C_{k-1}B$ in planar map geometry (M,μ) is defined by*

$$|AB| = |AC_1| + |C_1C_2| + |C_2C_3| + \cdots + |C_{k-1}B|.$$

As that shown in Chapter 6, there are not always exist a circle with any center and a given radius in planar map geometry in the usual sense of Euclid's definition. Since we have introduced angle function on planar map geometry, we can likewise the Euclid's definition to define an s-circle in planar map geometry.

Definition 7.4.2 *A closed curve C without self-intersection in planar map geometry (M,μ) is called an s-circle if there exists an s-point O in (M,μ) and a real number r such that $|OP| = r$ for each s-point P on C.*

Two Examples for s-circles in a planar map geometry (M,μ) are shown in Fig.7.4.1(1) and (2). The s-circle in Fig.7.4.1(1) is a circle in the Euclid's sense, but (2) is not. Notice that in Fig.7.4.1(2), s-points u and v are elliptic and the length $|OQ| = |Ou| + |uQ| = r$ for an s-point Q on the s-circle C, which seems likely an ellipse but it is not. The s-circle C in Fig.7.4.1(2) also implied that s-circles are more complex than those in Euclid plane geometry.

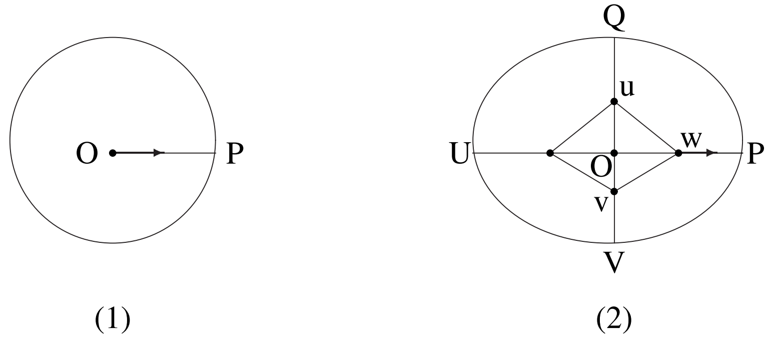


Fig.7.4.1

We know a necessary and sufficient condition for the existence of an s-circle in planar map geometry following.

Theorem 7.4.1 *Let (M,μ) be a planar map geometry on a plane $\sum$ and O an s-point on (M,μ). For a real number r, there is an s-circle of radius r with center O if and only if O is in the non-outer face or in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M in a Euclid plane $\sum$ are Euclidean points.*

Proof If there is a solitary non-Euclidean point A with $|OA| < r$, then by those materials in Chapter 3, there are no s-circles in (M,μ) of radius r with center O.

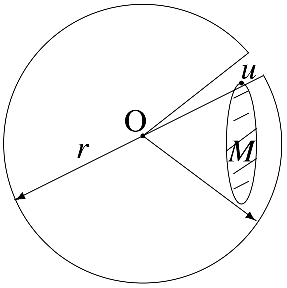
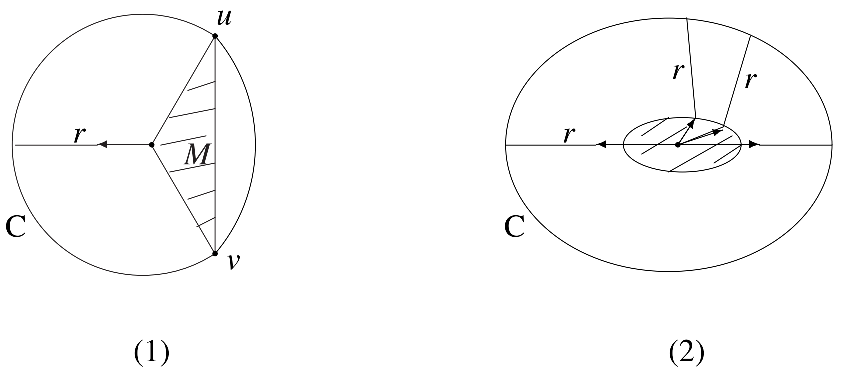


Fig.7.4.2

If O is in the outer face of M but there exists a number $\epsilon, r > \epsilon > 0$ such that one of the initial and final intersection points of a circle of radius ϵ with M on $\sum$ is non-Euclidean point, then points with distance r to O in (M,μ) at least has a gap in a circle with a Euclid sense. See Fig.7.4.2 for details, in where u is a non-Euclidean point and the shade field denotes the map M. Therefore there are no s-circles in (M,μ) of radius r with center O.

Now if O in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M on $\sum$ are Euclidean points or O is in a non-outer face of M, then by the definition of angle functions, we know that all points with distance r to O is a closed smooth curve on $\sum$, for example, see Fig.7.4.3(1) and (2).



(1) (2)

Fig.7.4.3

Whence it is an s-circle. □

We construct a polar axis OX with center O in planar map geometry as that in Euclid geometry. Then each s-point A has a coordinate (ρ, θ), where ρ is the length of the s-line segment OA and θ is the angle between OX and the straight line segment of OA containing the point A. We get an equation for an s-circle of radius r which has the same form as that in the analytic geometry of plane.

Theorem 7.4.2 *In a planar geometry* (M,μ) *with a polar axis OX of center O, the equation of each s-circle of radius r with center O is*

$$\rho = r.$$

Proof By the definition of s-circle C of radius r, every s-point on C has a distance r to its center O. Whence, its equation is $\rho = r$ in a planar map geometry with a polar axis OX of center O. □

§7.5 LINE BUNDLES IN PLANAR MAP GEOMETRY

7.5.1 Line Bundle. Among those s-line bundles the most important is parallel bundles defined in the next definition, motivated by the Euclid's fifth postulate.

Definition 7.5.1 *A family $\mathcal{L}$ of infinite s-lines not intersecting each other in planar geometry (M, μ) is called a parallel bundle.*

In Fig.7.5.1, we present all cases of parallel bundles passing through an edge in planar geometries, where, (a) is the case with the same type points u, v and $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$, (b) and (c) are the same type cases with $\rho_M(u)\mu(u) > \rho_M(v)\mu(v)$ or $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) > 2\pi$ or $< 2\pi$ and (d) is the case with an elliptic point u but a hyperbolic point v.

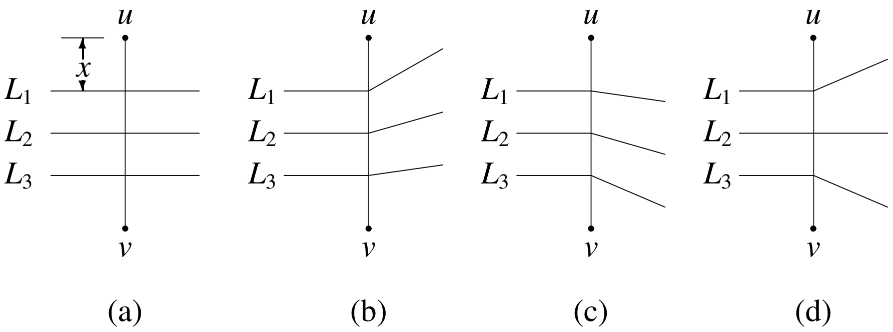


Fig.7.5.1

Here, we assume the angle at the intersection point is in clockwise, that is, a line passing through an elliptic point will bend up and passing through a hyperbolic point will bend down, such as those cases (b),(c) in the Fig.7.5.1. Generally, we define a *sign function sign(f) of an angle function f* as follows.

Definition 7.5.2 *For a vector $\overrightarrow{O}$ on the Euclid plane called an orientation, a sign function sign(f) of an angle function f at an s-point u is defined by*

$$sign(f)(u) = \begin{cases} 1, & \text{if } u \text{ is elliptic,} \\ 0, & \text{if } u \text{ is euclidean,} \\ -1, & \text{if } u \text{ is hyperbolic.} \end{cases}$$

We classify parallel bundles in planar map geometry along an orientation $\overrightarrow{O}$ in this section.

7.5.2 Necessary and Sufficient Condition for Parallel Bundle. We investigate the behaviors of parallel bundles in planar map geometry (M,μ). Denote by $f(x)$ the angle function value at an intersection s-point of an s-line L with an edge (u,v) of M and a distance x to u on (u,v) as shown in Fig.7.5.1(a). Then

Theorem 7.5.1 *A family $\mathcal{L}$ of parallel s-lines passing through an edge (u,v) is a parallel bundle if and only if*

$$\left.\frac{df}{dx}\right|_{+} \geq 0.$$

Proof If $\mathcal{L}$ is a parallel bundle, then any two s-lines L_1, L_2 will not intersect after them passing through the edge uv. Therefore, if θ_1,θ_2 are the angles of L_1, L_2 at the intersection s-points of L_1, L_2 with (u,v) and L_2 is far from u than L_1, then we know $\theta_2 \geq \theta_1$. Thereby we know that $f(x+\Delta x)-f(x) \geq 0$ for any point with distance x from u and $\Delta x > 0$. Therefore, we get that

$$\left.\frac{df}{dx}\right|_{+} = \lim_{\Delta x\to +0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \geq 0.$$

As that shown in the Fig.7.5.1.

Now if $\left.\frac{df}{dx}\right|_{+} \geq 0$, then $f(y) \geq f(x)$ if $y \geq x$. Since $\mathcal{L}$ is a family of parallel s-lines before meeting uv, any two s-lines in $\mathcal{L}$ will not intersect each other after them passing through (u,v). Therefore, $\mathcal{L}$ is a parallel bundle. □

A general condition for a family of parallel s-lines passing through a cut of a planar map being a parallel bundle is the following.

Theorem 7.5.2 *Let (M,μ) be a planar map geometry, $C = \{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2,\cdots,f_l$ (also seeing Fig.7.5.2), respectively.*

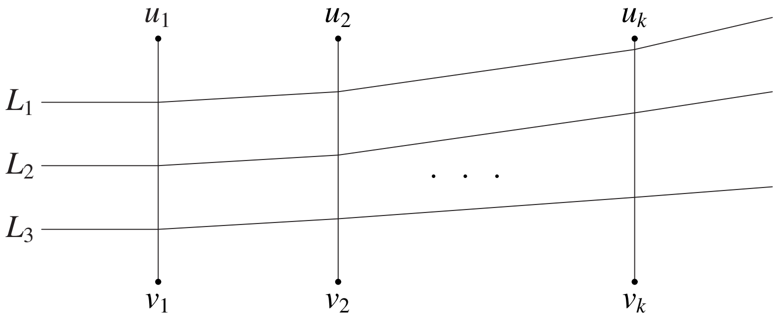


Fig.7.5.2

Then a family $\mathcal{L}$ of parallel s-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$,

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0,\\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0,\\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) &\geq 0,\\
\cdots\cdots\cdots\cdots&,\\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) &\geq 0.
\end{aligned}$$

Proof According to Theorem 7.5.1, we know that s-lines will not intersect after them passing through (u_1, v_1) and (u_2, v_2) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_2)(x)f_2(x + \Delta x) + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_2)(x)f_2(x),$$

seeing Fig.7.5.3 for an explanation.

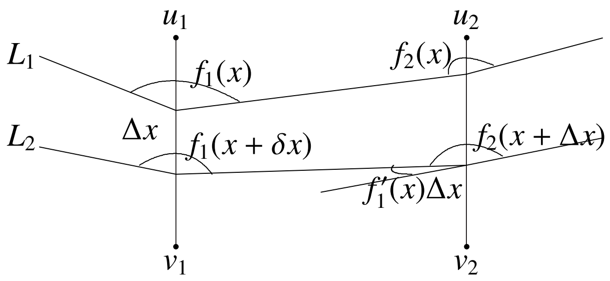


Fig.7.5.3

That is,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0.$$

Similarly, s-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2)$ and (u_3, v_3) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$\begin{aligned}
sign(f_3)(x)f_3(x + \Delta x) \quad + \quad & sign(f_2)(x)f'_{2+}(x)\Delta x\\
& + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_3)(x)f_3(x).
\end{aligned}$$

Namely,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) \geq 0.$$

Generally, the s-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2), \cdots,$ (u_{l-1}, v_{l-1}) and (u_l, v_l) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$\begin{aligned} sign(f_l)(x)f_l(x + \Delta x) \quad + \quad & sign(f_{l-1})(x)f'_{l-1+}(x)\Delta x + \\ & \cdots + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_l)(x)f_l(x). \end{aligned}$$

Whence, we get that

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel s-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$, we have that

$$\begin{aligned} sign(f_1)(x)f'_{1+}(x) \geq 0, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) \geq 0, \\ \cdots\cdots\cdots\cdots, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) \geq 0. \end{aligned}$$

This completes the proof. □

Corollary 7.5.1 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then a family $\mathcal{L}$ of parallel lines passing through C is still parallel lines after them leaving C if and only if for any $x, x \geq 0$,*

$$\begin{aligned} sign(f_1)(x)f'_{1+}(x) \geq 0, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) \geq 0, \\ \cdots\cdots\cdots\cdots, \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) \geq 0. \end{aligned}$$

and

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) = 0.$$

Proof According to Theorem 7.5.2, we know the condition is a necessary and sufficient condition for $\mathcal{L}$ being a parallel bundle. Now since lines in $\mathcal{L}$ are parallel lines after them leaving C if and only if for any $x \geq 0$ and $\Delta x \geq 0$, there must be that

$$sign(f_l)f_l(x+\Delta x) + sign(f_{l-1})f'_{l-1+}(x)\Delta x + \cdots + sign(f_1)f'_{1+}(x)\Delta x = sign(f_l)f_l(x).$$

Therefore, we get that

$$sign(f_1)(x)f'_{1+}(x)+sign(f_2)(x)f'_{2+}(x)+\cdots+sign(f_1)(x)f'_{l+}(x) = 0. \qquad \square$$

There is a natural question on parallel bundles in planar map geometry. That is *when do some parallel s-lines parallel the initial parallel lines after them passing through a cut C in a planar map geometry?* The answer is the next result.

Theorem 7.5.3 *Let (M,μ) be a planar map geometry, $C = \{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then the parallel s-lines parallel the initial parallel lines after them passing through C if and only if for $\forall x \geq 0$,*

$$\begin{aligned} sign(f_1)(x)f'_{1+}(x) &\geq 0,\\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0,\\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0,\\ \cdots\cdots\cdots\cdots&,\\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) &\geq 0. \end{aligned}$$

and

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_1)(x)f_l(x) = l\pi.$$

Proof According to Theorem 7.5.2 and Corollary 7.5.1, we know that these parallel s-lines satisfying conditions of this theorem is a parallel bundle.

We calculate the angle $\alpha(i,x)$ of an s-line L passing through an edge u_iv_i, $1 \leq i \leq l$ with the line before it meeting C at the intersection of L with the edge (u_i,v_i), where x is the distance of the intersection point to u_1 on (u_1,v_1), see also Fig.4.18. By definition, we know the angle $\alpha(1,x) = sign(f_1)f(x)$ and $\alpha(2,x) = sign(f_2)f_2(x) - (\pi - sign(f_1)f_1(x)) = sign(f_1)f_1(x) + sign(f_2)f_2(x) - \pi$.

Now if $\alpha(i,x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_i)f_i(x) - (i-1)\pi$, then we know that $\alpha(i+1,x) = sign(f_{i+1})f_{i+1}(x) - (\pi - \alpha(i,x)) = sign(f_{i+1})f_{i+1}(x) + \alpha(i,x) - \pi$ similar to the case $i = 2$. Thereby we get that

$$\alpha(i+1,x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_{i+1})f_{i+1}(x) - i\pi.$$

Notice that an s-line L parallel the initial parallel line after it passing through C if and only if $\alpha(l,x) = \pi$, i.e.,

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_l)f_l(x) = l\pi.$$

This completes the proof. $\square$

7.5.3 Linear Conditions for Parallel Bundle. For the simplicity, we can assume even that the function $f(x)$ is linear and denoted it by $f_l(x)$. We calculate $f_l(x)$ in the first.

Theorem 7.5.4 *The angle function $f_l(x)$ of an s-line L passing through an edge (u,v) at a point with distance x to u is*

$$f_l(x) = \left(1 - \frac{x}{d(u,v)}\right)\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2},$$

where, $d(u,v)$ is the length of the edge (u,v).

Proof Since $f_l(x)$ is linear, we know that $f_l(x)$ satisfies the following equation.

$$\frac{f_l(x) - \dfrac{\rho(u)\mu(u)}{2}}{\dfrac{\rho(v)\mu(v)}{2} - \dfrac{\rho(u)\mu(u)}{2}} = \frac{x}{d(u,v)},$$

Calculation shows that

$$f_l(x) = \left(1 - \frac{x}{d(u,v)}\right)\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2}. \qquad \square$$

Corollary 7.5.2 *Under the linear assumption, a family $\mathcal{L}$ of parallel s-lines passing through an edge (u,v) is a parallel bundle if and only if*

$$\frac{\rho(u)}{\rho(v)} \le \frac{\mu(v)}{\mu(u)}.$$

Proof According to Theorem 7.5.1, a family of parallel s-lines passing through an edge (u,v) is a parallel bundle if and only if $f'(x) \ge 0$ for $\forall x, x \ge 0$, i.e.,

$$\frac{\rho(v)\mu(v)}{2d(u,v)} - \frac{\rho(u)\mu(u)}{2d(u,v)} \ge 0.$$

Therefore, a family $\mathcal{L}$ of parallel s-lines passing through an edge (u, v) is a parallel bundle if and only if

$$\rho(v)\mu(v) \geq \rho(u)\mu(u).$$

Whence,

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}. \qquad \square$$

For a family of parallel s-lines passing through a cut, we get the following result.

Theorem 7.5.5 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$. Then under the linear assumption, a family L of parallel s-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system*

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1),$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1, v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2, v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1, v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2, v_2)},$$

$$\cdots\cdots\cdots\cdots,$$

$$\begin{aligned} \frac{\rho(v_1)\mu(v_1)}{d(u_1, v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2, v_2)} + \cdots + \frac{\rho(v_l)\mu(v_l)}{d(u_l, v_l)} \\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1, v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2, v_2)} + \cdots + \frac{\rho(u_l)\mu(u_l)}{d(u_l, v_l)}. \end{aligned}$$

Proof Under the linear assumption, for any integer $i, i \geq 1$ we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_i, v_i)}$$

by Theorem 7.5.4. Thereby, according to Theorem 7.5.2, we get that a family L of parallel s-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1),$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1, v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2, v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1, v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2, v_2)},$$

$$\cdots\cdots\cdots\cdots,$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(v_l)\mu(v_l)}{d(u_l,v_l)}\\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

This completes the proof. □

For planar maps underlying a regular graph, we have an interesting consequence for parallel bundles in the following.

Corollary 7.5.3 *Let (M,μ) be a planar map geometry with M underlying a regular graph, $C=\{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l\geq 1$. Then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system.*

$$\mu(v_1)\geq\mu(u_1),$$

$$\frac{\mu(v_1)}{d(u_1,v_1)}+\frac{\mu(v_2)}{d(u_2,v_2)}\geq\frac{\mu(u_1)}{d(u_1,v_1)}+\frac{\mu(u_2)}{d(u_2,v_2)},$$

$$\cdots\cdots\cdots\cdots,$$

$$\frac{\mu(v_1)}{d(u_1,v_1)}+\frac{\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\mu(v_l)}{d(u_l,v_l)}\geq\frac{\mu(u_1)}{d(u_1,v_1)}+\frac{\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\mu(u_l)}{d(u_l,v_l)}$$

and particularly, if assume that all the lengths of edges in C are the same, then

$$\begin{aligned}\mu(v_1) &\geq \mu(u_1)\\ \mu(v_1)+\mu(v_2) &\geq \mu(u_1)+\mu(u_2)\\ \cdots\cdots &\cdots \cdots\cdots\\ \mu(v_1)+\mu(v_2)+\cdots+\mu(v_l) &\geq \mu(u_1)+\mu(u_2)+\cdots+\mu(u_l).\end{aligned}$$

Certainly, by choice different angle factors we can also get combinatorial conditions for the existence of parallel bundles under the linear assumption.

Theorem 7.5.6 *Let (M,μ) be a planar map geometry, $C=\{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l\geq 1$. If*

$$\frac{\rho(u_i)}{\rho(v_i)}\leq\frac{\mu(v_i)}{\mu(u_i)}$$

for any integer $i, i\geq 1$, then a family L of parallel s-lines passing through C is a parallel bundle under the linear assumption.

Proof Under the linear assumption we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_i, v_i)}$$

for any integer $i, i \geq 1$ by Theorem 7.5.4. Thereby $f'_{i+}(x) \geq 0$ for $i = 1, 2, \cdots, l$. We get that

$$\begin{aligned} f'_1(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) &\geq 0 \\ f'_{1+}(x) + f'_{2+}(x) + f'_{3+}(x) &\geq 0 \\ \cdots\cdots\cdots\cdots& \\ f'_{1+}(x) + f'_{2+}(x) + \cdots + f'_{l+}(x) &\geq 0. \end{aligned}$$

By Theorem 7.5.2 we know that a family L of parallel s-lines passing through C is still a parallel bundle. □

§7.6 **EXAMPLES OF PLANAR MAP GEOMETRY**

By choice different planar maps and angle factors on their vertices, we can get various planar map geometries. In this section, we present some concrete examples for planar map geometry.

Example 7.6.1 *A complete planar map* K_4.

We take a complete map K_4 embedded on the plane $\sum$ with vertices u, v, w and t and angle factors

$$\mu(u) = \frac{\pi}{2},\ \ \mu(v) = \mu(w) = \pi \text{ and } \mu(t) = \frac{2\pi}{3},$$

such as shown in Fig.7.6.1 where each number on the side of a vertex denotes $\rho_M(x)\mu(x)$ for $x = u, v, w$ and t. Assume the linear assumption is holds in this planar map geometry (M, μ). Then we get a classifications for s-points in (M, μ) as follows.

$$V_{el} = \{\text{points in } (\text{uA} \setminus \{\text{A}\}) \bigcup (\text{uB} \setminus \{\text{B}\}) \bigcup (\text{ut} \setminus \{\text{t}\})\},$$

where A and B are Euclidean points on (u, w) and (u, v), respectively.

$$V_{eu} = \{A, B, t\} \bigcup (P \setminus E(K_4))$$

and

$$V_{hy} = \{\text{points in } (\mathrm{wA} \setminus \{\mathrm{A}\}) \bigcup (\mathrm{wt} \setminus \{\mathrm{t}\}) \bigcup \mathrm{wv} \bigcup (\mathrm{tv} \setminus \{\mathrm{t}\}) \bigcup (\mathrm{vB} \setminus \{\mathrm{B}\})\}.$$

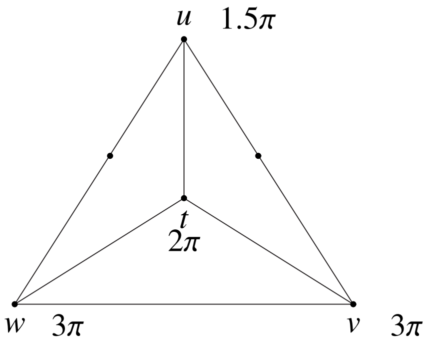


Fig.7.6.1

We assume that the linear assumption holds in this planar map geometry (M, μ). Then we get a classifications for s-points in (M, μ) as follows.

$$V_{el} = \{\text{points in } (\mathrm{uA} \setminus \{\mathrm{A}\}) \bigcup (\mathrm{uB} \setminus \{\mathrm{B}\}) \bigcup (\mathrm{ut} \setminus \{\mathrm{t}\})\},$$

where A and B are Euclidean points on (u, w) and (u, v), respectively.

$$V_{eu} = \{A, B, t\} \bigcup (P \setminus E(K_4))$$

$$V_{hy} = \{\text{points in } (\mathrm{wA} \setminus \{\mathrm{A}\}) \bigcup (\mathrm{wt} \setminus \{\mathrm{t}\}) \bigcup \mathrm{wv} \bigcup (\mathrm{tv} \setminus \{\mathrm{t}\}) \bigcup (\mathrm{vB} \setminus \{\mathrm{B}\})\}.$$

Edges in K_4 are classified into $(u, t) \in C_E^1, (t, w), (t, v) \in C_E^3$, $(u, w), (u, v) \in C_E^5$ and $(w, u) \in C_E^6$. Various s-lines in this planar map geometry are shown in Fig.7.6.2 following.

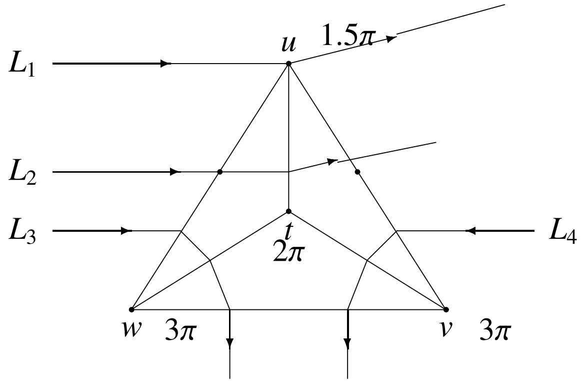


Fig.7.6.2

There are no 1-polygons in this planar map geometry. One 2-polygon and various triangles are shown in Fig.7.6.3.

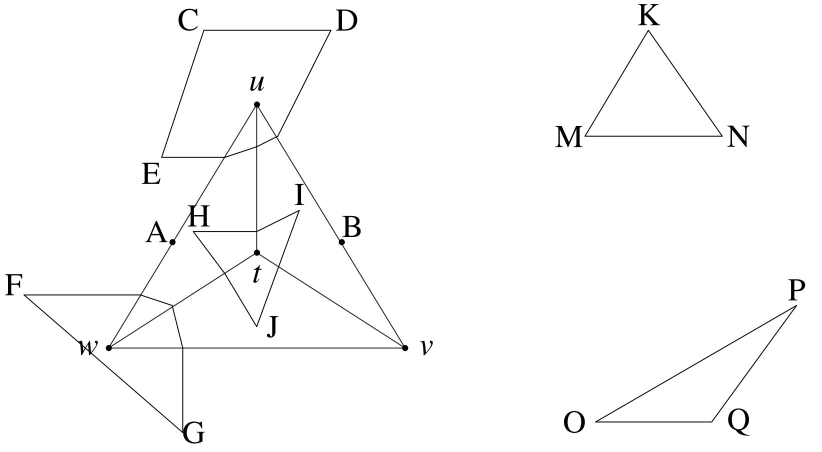


Fig.7.6.3

Example 7.6.2 *A wheel planar map $W_{1.4}$.*

We take a wheel $W_{1.4}$ embedded on a plane $\sum$ with vertices O and u, v, w, t and angle factors

$$\mu(O) = \frac{\pi}{2}, \text{ and } \mu(u) = \mu(v) = \mu(w) = \mu(t) = \frac{4\pi}{3},$$

such as shown in Fig.7.6.4.

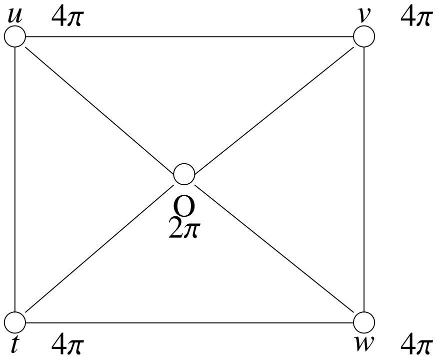


Fig.7.6.4

There are no elliptic points in this planar map geometries. Euclidean and hyperbolic points V_{eu}, V_{hy} are as follows.

$$V_{eu} = P\bigcup\backslash(E(W_{1.4})\setminus\{O\})$$

and

$$V_{hy} = E(W_{1.4}) \setminus \{O\}.$$

Edges are classified into $(O, u), (O, v), (O, w), (O, t) \in C_E^3$ and $(u, v), (v, w), (w, t), (t, u) \in C_E^6$. Various s-lines and one 1-polygon are shown in Fig.7.6.5 where each s-line will turn to its opposite direction after it meeting $W_{1.4}$ such as those s-lines L_1, L_2 and L_4, L_5 in Fig.7.6.5.

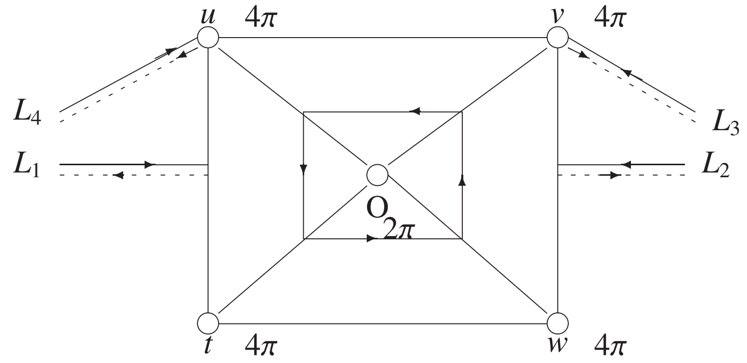


Fig.7.6.5

Example 7.6.3 *A parallel bundle in a planar map geometry.*

We choose a planar ladder and define its angle factor as shown in Fig.7.6.6 where each number on the side of a vertex u denotes the number $\rho_M(u)\mu(u)$. Then we find a parallel bundle $\{L_i; 1 \le i \le 6\}$ as those shown in Fig.7.6.6.

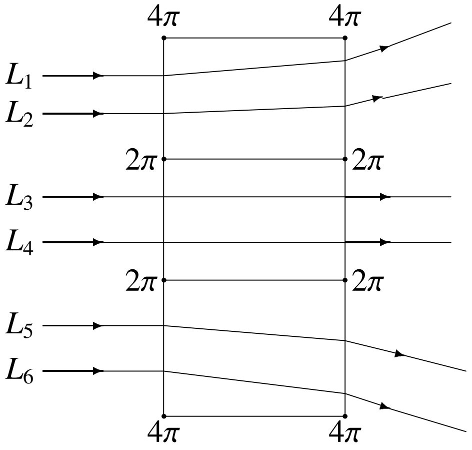


Fig.7.6.6

§7.7 RESEARCH PROBLEMS

7.7.1. As a generalization of Euclid geometry of plane by Smarandache's notion, the planar map geometry was introduced in [Mao8] and discussed in [Mao8]-[Mao11], [Mao17]. Similarly, a generalization can be also done for Euclid geometry of space $\mathbf{R}^3$. Some open problems on this generalization are listed following.

Problem 7.7.1 *Establish Smarandache geometry by embedded graphs in space* $\mathbf{R}^3$ *and classify their fundamental elements, such as those of points, lines, polyhedrons,* $\cdots$*, etc..*

Problem 7.7.2 *Determine various surfaces and convex polyhedrons in Smarandache geometry of space* $\mathbf{R}^3$*, such as those of sphere, surface of cylinder, circular cone, torus, double torus, projective plane, Klein bottle and tetrahedron, pentahedron, hexahedron,* $\cdots$*, etc..*

Problem 7.7.3 *Define the conception of volume in Smarandache geometry on space* $\mathbf{R}^3$ *and find formulae for volumes of convex polyhedrons, such as those of tetrahedron, pentahedron or hexahedron,* $\cdots$*, etc..*

Problem 7.7.4 *Apply s-lines in Smarandache geometry of space* $\mathbf{R}^3$ *to knots and find new characteristics.*

7.7.2 As pointed out in last chapter, we can also establish map geometry on locally orientable surfaces and find its fundamental elements of points, lines, polyhedrons, $\cdots$, etc., particularly, on sphere, torus, double torus, projective plane, Klein bottle, $\cdots$, i.e., to establish an intrinsic geometry on surface. For this objective, open problems for such surfaces with small genus should be considered first.

Problem 7.7.5 *Establish an intrinsic geometry by map geometry on sphere or torus and find its fundamental elements.*

Problem 7.7.6 *Establish an intrinsic geometry on projective or Klein bottle and find its fundamental elements.*

Problem 7.7.7 *Define various measures of map geometry on a locally orientable surface* S *and apply them to characterize the surface* S.

Problem 7.7.8 *Define the conception of curvature for map geometry* (M,μ) *on locally orientable surfaces and calculate the sum* $\omega(M)$ *of curvatures on all edges in* M.

We have a conjecture following, which generalizes the Gauss-Bonnet theorem.

Conjecture 7.7.1 *$\omega(M) = 2\pi\chi(M)s(M)$, where $s(M)$ denotes the sum of length of edges in M.*

7.7.3 It should be noted that nearly all branches of physics apply Euclid space $\mathbf{R}^3$ to a spacetime for its concise and homogeneity unless Einstein's relativity theory. This has their own reason, also due to one's observation because the moving of particle is more likely that in Euclid space $\mathbf{R}^3$. However, as shown in relativity theory, this realization is incorrect in general for the real world is hybridization and not homogenous. That is why a physical theory on $\mathbf{R}^3$ can only find unilateral behavior of particles.

Problem 7.7.9 *Establish a suitable spacetime by space $\mathbf{R}^3$ in Smarandache geometry with time axis t and find the global behaviors of particles.*

Problem 7.7.10 *Establish a unified theory of mechanics, thermodynamics, optics, electricity, $\cdots$, etc. by that of Smarandachely spacetime such that each of these theory is its a case.*

CHAPTER 8.

Pseudo-Euclidean Geometry

The essential idea in planar map geometry is associating each point in a planar map with an angle factor, which turns flatness of a plane to tortuous. When the order of a planar map tends to infinite and its diameter of each face tends to zero (such planar maps naturally exist, for example, planar triangulations), we get a tortuous plane at the limiting point, i.e., a plane equipped with a vector and straight lines maybe not exist. Such a consideration can be applied to Euclidean spaces and manifolds. We discuss them in this chapter. Sections 8.1-8.3 concentrate on pseudo-planes with curve equations, integral curves and stability of differential equations. The pseudo-Euclidean geometry on $\mathbf{R}^n$ for $n \geq 3$ is introduced in Section 8.4, in where conditions for a curve existed in such a pseudo-Euclidean space and the representation for angle function by rotation matrix are found. Particularly, the finitely pseudo-Euclidean geometry is characterized by graphs embedded in $\mathbf{R}^n$. The Section 8.5 can be viewed as an elementary introduction to smooth pseudo-manifold, i.e., differential pseudo-manifolds. Further consideration on this topics will establish the application of pseudo-manifolds to physics (see [Mao33] or [Mao38] for details).

§8.1 PSEUDO-PLANES

8.1.1 Pseudo-Plane. In the classical analytic geometry on plane, each point is correspondent with a Descartes coordinate (x, y), where x and y are real numbers which ensures the flatness of a plane. Motivated by the ideas in Chapters 6-7, we find a new kind of plane, called *pseudo-plane*, which distort the flatness of a plane and can be applied to sciences.

Definition 8.1.1 *Let $\sum$ be a Euclid plane. For $\forall u \in \sum$, if there is a continuous mapping $\omega : u \to \omega(u)$ where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any chosen number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in \sum$, $\|u-v\| < \delta$ such that $\|\omega(u)-\omega(v)\| < \epsilon$, then $\sum$ is called a pseudo-plane, denoted by $(\sum, \omega)$, where $\|u - v\|$ denotes the norm between points u and v in $\sum$.*

An explanation for Definition 8.1.1 is shown in Fig.8.1.1, in where $n = 1$ and $\omega(u)$ is an angle function $\forall u \in \sum$.

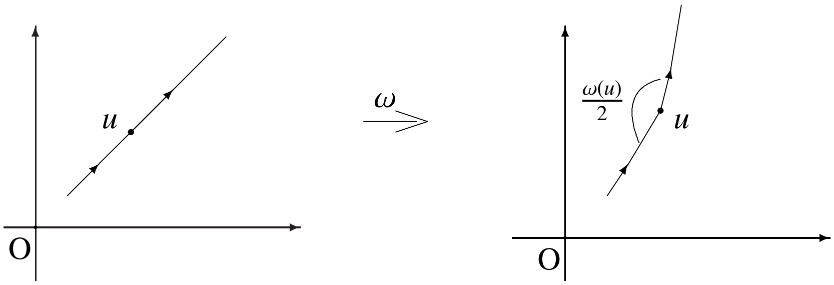


Fig.8.1.1

We can also explain $\omega(u)$, $u \in \sum$ to be the coordinate z in $u = (x, y, z) \in \mathbf{R}^3$ by taking also $n = 1$. Thereby a pseudo-plane can be viewed as a projection of a Euclid space $\mathbf{R}^{n+2}$ on a Euclid plane. This fact implies that some characteristic of the geometry on space may reflected by a pseudo-plane.

We only discuss the case of $n = 1$ and explain $\omega(u), u \in \sum$ being a periodic function in this chapter, i.e., for any integer k, $4k\pi + \omega(u) \equiv \omega(u)(mod\ 4\pi)$. Not loss of generality, we assume that $0 < \omega(u) \leq 4\pi$ for $\forall u \in \sum$. Similar to map geometry, points in a pseudo-plane are classified into three classes, i.e., *elliptic points V_{el}*, *Euclidean points V_{eu}* and *hyperbolic points V_{hy}*, defined respectively by

$$V_{el} = \left\{u \in \sum |\omega(u) \ < 2\pi\right\},$$

$$V_{eu} = \left\{v \in \sum |\omega(v) \ = 2\pi\right\}$$

and

$$V_{hy} = \left\{w \in \sum |\omega(w) \ > 2\pi\right\}.$$

Then we get the following result.

Theorem 8.1.1 *There is a straight line segment AB in a pseudo-plane* $(\sum, \omega)$ *if and only if for* $\forall u \in AB$, $\omega(u) = 2\pi$, *i.e., every point on AB is Euclidean.*

Proof Since $\omega(u)$ is an angle function for $\forall u \in \sum$, we know that AB is a straight line segment if and only if for $\forall u \in AB$, $\frac{\omega(u)}{2} = \pi$. Thus $\omega(u) = 2\pi$ and u is Euclidean. □

Theorem 8.1.1 implies that there maybe no straight line segments in a pseudo-plane.

Corollary 8.1.1 *If there are only finite Euclidean points in a pseudo-plane* $(\sum, \omega)$*, then there are no straight line segments in* $(\sum, \omega)$.

Corollary 8.1.2 *There are not always exist a straight line between two given points u and v in a pseudo-plane* $(\sum, \omega)$.

By the intermediate value theorem in calculus, we get the following result for points in pseudo-planes.

Theorem 8.1.2 *In a pseudo-plane* $(\sum, \omega)$*, if* $V_{el} \neq \emptyset$ *and* $V_{hy} \neq \emptyset$*, then* $V_{eu} \neq \emptyset$.

Proof By these assumptions, we can choose points $u \in V_{el}$ and $v \in V_{hy}$. Consider points on line segment uv in a Euclid plane $\sum$. Since $\omega(u) < 2\pi$ and $\omega(v) > 2\pi$, there exists at least a point $w, w \in uv$ such that $\omega(w) = 2\pi$, i.e., $w \in V_{eu}$ by the intermediate value theorem in calculus. Whence, $V_{eu} \neq \emptyset$. □

Corollary 8.1.3 *In a pseudo-plane* $(\sum, \omega)$*, if* $V_{eu} = \emptyset$*, then every point of* $(\sum, \omega)$ *is elliptic or every point of* $\sum$ *is hyperbolic.*

According to Corollary 8.1.3, we classify pseudo-planes into four classes following.

C_P^1(**Euclidean**): *pseudo-planes whose each point is Euclidean.*
C_P^2(**elliptic**): *pseudo-planes whose each point is elliptic.*
C_P^3(**hyperbolic**): *pseudo-planes whose each point is hyperbolic.*
C_P^4(**Smarandachely**): *pseudo-planes in which there are Euclidean, elliptic and hyperbolic points simultaneously.*

8.1.2 Curve Equation. We define the *sign function* sign(v) on point in a pseudo-plane $(\sum, \omega)$ by

$$\operatorname{sign}(v) = \begin{cases} 1, & \text{if } v \text{ is elliptic,} \\ 0, & \text{if } v \text{ is euclidean,} \\ -1, & \text{if } v \text{ is hyperbolic.} \end{cases}$$

Then we get a criteria following for the existence of an algebraic curve C in pseudo-plane $(\sum, \omega)$.

Theorem 8.1.3 *There is an algebraic curve* $F(x, y) = 0$ *passing through* (x_0, y_0) *in a domain* D *of pseudo-plane* $(\sum, \omega)$ *with Descartes coordinate system if and only if* $F(x_0, y_0) = 0$ *and for* $\forall(x, y) \in D$,

$$\left(\pi - \frac{\omega(x, y)}{2}\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \operatorname{sign}(x, y).$$

Proof By the definition of pseudo-planes in the case of that ω being an angle function and the geometrical meaning of differential value, such as those shown in Fig.8.1.2 following,

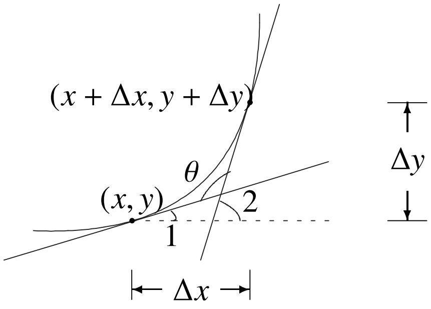


Fig.8.1.2

where $\theta = \pi - \angle 2 + \angle 1$, $\lim\limits_{\Delta x \to 0} \theta = \omega(x, y)$ and (x, y) is an elliptic point, we know that an algebraic curve $F(x, y) = 0$ exists in a domain D of $(\sum, \omega)$ if and only if

$$\left(\pi - \frac{\omega(x, y)}{2}\right) = \operatorname{sign}(x, y)\frac{d(\arctan(\frac{dy}{dx})}{dx},$$

for $\forall(x, y) \in D$, i.e.,

$$\left(\pi - \frac{\omega(x, y)}{2}\right) = \frac{\operatorname{sign}(x, y)}{1 + (\frac{dy}{dx})^2}.$$

Therefore,

$$\left(\pi - \frac{\omega(x,y)}{2}\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \mathrm{sign}(x,y). \qquad \square$$

A plane curve C is called *elliptic* or *hyperbolic* if $sign(x,y) = 1$ or -1 for each point (x,y) on C. We get a conclusion for the existence of elliptic or hyperbolic curves in a pseudo-plane by Theorem 8.1.3 following.

Corollary 8.1.4 *An elliptic curve $F(x,y) = 0$ exists in pseudo-plane $(\sum, \omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $D \subset \sum$ such that $F(x_0,y_0) = 0$ and for $\forall (x,y) \in D$,*

$$\left(\pi - \frac{\omega(x,y)}{2}\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 1.$$

Similarly, there exists a hyperbolic curve $H(x,y) = 0$ in a pseudo-plane $(\sum, \omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $U \subset \sum$ such that for $H(x_0,y_0) = 0$ and $\forall (x,y) \in U$,

$$\left(\pi - \frac{\omega(x,y)}{2}\right)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = -1.$$

Construct a polar axis (ρ,θ) in pseudo-plane $(\sum, \omega)$. We get a result following.

Theorem 8.1.4 *There is an algebraic curve $f(\rho,\theta) = 0$ passing through (ρ_0,θ_0) in a domain F of pseudo-plane $(\sum, \omega)$ with polar coordinate system if and only if $f(\rho_0,\theta_0) = 0$ and for $\forall (\rho,\theta) \in F$,*

$$\pi - \frac{\omega(\rho,\theta)}{2} = \mathrm{sign}(\rho,\theta)\frac{d\theta}{d\rho}.$$

Proof Similar to that proof of Theorem 8.1.3, we know that $\lim\limits_{\triangle x \to 0} \theta = \omega(x,y)$ and $\theta = \pi - \angle 2 + \angle 1$ if (ρ,θ) is elliptic, or $\theta = \pi - \angle 1 + \angle 2$ if (ρ,θ) is hyperbolic in Fig.8.1.2. Consequently, we get that

$$\pi - \frac{\omega(\rho,\theta)}{2} = \mathrm{sign}(\rho,\theta)\frac{d\theta}{d\rho}. \qquad \square$$

Corollary 8.1.5 *An elliptic curve $F(\rho,\theta) = 0$ exists in pseudo-plane $(\sum, \omega)$ with polar coordinate system passing through (ρ_0,θ_0) if and only if there is a domain $F \subset \sum$ such that $F(\rho_0,\theta_0) = 0$ and for $\forall (\rho,\theta) \in F$,*

$$\pi - \frac{\omega(\rho,\theta)}{2} = \frac{d\theta}{d\rho},$$

and there exists a hyperbolic curve $h(x,y)=0$ *in pseudo-plane* $(\sum,\omega)$ *with polar coordinate system passing through* (ρ_0,θ_0) *if and only if there is a domain* $U\subset\sum$ *such that* $h(\rho_0,\theta_0)=0$ *and for* $\forall(\rho,\theta)\in U$,

$$\pi-\frac{\omega(\rho,\theta)}{2}=-\frac{d\theta}{d\rho}.$$

8.1.3 Planar Presented $\mathbf{R}^3$. We discuss a presentation for points in $\mathbf{R}^3$ by the Euclid plane $\mathbf{R}^2$ with characteristics.

Definition 8.1.2 *For a point* $P=(x,y,z)\in\mathbf{R}^3$ *with center O, let* ϑ *be the angle of vector* $\overrightarrow{OP}$ *with the plane XOY. Define an angle function* $\omega:(x,y)\rightarrow 2(\pi-\vartheta)$, *i.e., the presentation of point* (x,y,z) *in* $\mathbf{R}^3$ *is a point* (x,y) *with* $\omega(x,y)=2(\pi-\angle(\overrightarrow{OP},XOY))$ *in pseudo-plane* $(\sum,\omega)$.

An explanation for Definition 8.1.1 is shown in Fig.8.1.3, where θ is an angle between the vector $\overrightarrow{OP}$ and plane XOY.

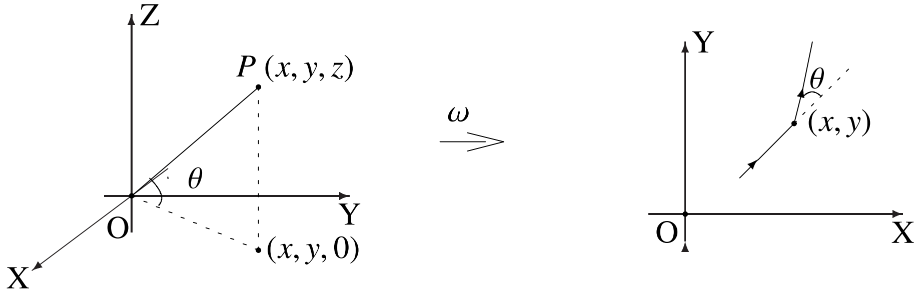


Fig.8.1.3

Theorem 8.1.5 *Let* $(\sum,\omega)$ *be a pseudo-plane and* $P=(x,y,z)$ *a point in* $\mathbf{R}^3$. *Then the point* (x,y) *is elliptic, Euclidean or hyperbolic if and only if* $z>0$, $z=0$ *or* $z<0$.

Proof By Definition 8.1.2, we know that $\omega(x,y)>2\pi$, $=2\pi$ or $<2\pi$ if and only if $\theta>0$,$=0$ or <0 by $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, which are equivalent to that $z>0$, $=0$ or <0. □

The following result brings light for the shape of points in $\mathbf{R}^3$ to that of points with a constant angle function value in pseudo-plane $(\sum,\omega)$.

Theorem 8.1.6 *For a constant* $\eta, 0<\eta\le 4\pi$, *all points* (x,y,z) *with* $\omega(x,y)=\eta$ *in* $\mathbf{R}^3$ *consist an infinite circular cone with vertex O and an angle* $\pi-\dfrac{\eta}{2}$ *between its generatrix and the plane XOY.*

Proof Notice that $\omega(x_1, y_1) = \omega(x_2, y_2)$ for two points A, B in $\mathbf{R}^3$ with $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ if and only if

$$\angle(\overrightarrow{OA}, XOY) = \angle(\overrightarrow{OB}, XOY) = \pi - \frac{\eta}{2},$$

thus points A and B are on a circular cone with vertex O and an angle $\pi - \frac{\eta}{2}$ between $\overrightarrow{OA}$ or $\overrightarrow{OB}$ and the plane XOY. Since $z \to +\infty$, we get an infinite circular cone in $\mathbf{R}^3$ with vertex O and an angle $\pi - \dfrac{\eta}{2}$ between its generatrix and the plane XOY. □

§8.2 INTEGRAL CURVES

8.2.1 Integral Curve. An *integral curve* in Euclid plane is defined by the definition following.

Definition 8.2.1 If the solution of a differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ *exists, then all points* (x, y) *consisted by their solutions of this initial problem on Euclid plane* $\sum$ *is called an integral curve.*

In the theory of ordinary differential equation, a well-known result for the unique solution of an ordinary differential equation is stated in the following.

Theorem 8.2.1 *If the following conditions hold:*

(1) $f(x, y)$ *is continuous in a field* F*:*

$$F : x_0 - a \le x \le x_0 + a, \quad y_0 - b \le y \le y_0 + b.$$

(2) *There exist a constant* ς *such that for* $\forall (x, y), (x, \overline{y}) \in F$,

$$|f(x, y) - f(x, \overline{y})| \le \varsigma |y - \overline{y}|,$$

then there is an unique solution $y = \varphi(x)$, $\varphi(x_0) = y_0$ *for the differential equation*

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ *in the interval* $[x_0 - h_0, x_0 + h_0]$, *where* $h_0 = \min\left(a, \dfrac{b}{M}\right)$ *with* $M = \max\limits_{(x,y)\in R} |f(x, y)|$.

A complete proof of Theorem 8.2.1 can be found in textbook on ordinary differential equations, such as the reference [Arn1]. It should be noted that conditions in Theorem 6.2.1 are complex and can not be applied conveniently. As we have shown in Section 8.1.1, a pseudo-plane $(\sum,\omega)$ is related with differential equations in Euclid plane $\sum$. Whence, by a geometrical view, to find an integral curve in pseudo-plane $(\sum,\omega)$ is equivalent to solve an initial problem for an ordinary differential equation. Thereby we concentrate on to find integral curves in pseudo-plane in this section.

According to Theorem 8.1.3, we get the following result.

Theorem 8.2.2 *A curve C,*

$$C=\left\{(x,y(x))|\frac{dy}{dx}=f(x,y),y(x_0)=y_0\right\}$$

exists in pseudo-plane $(\sum,\omega)$ *if and only if there is an interval* $I=[x_0-h,x_0+h]$ *and an angle function* $\omega:\sum\rightarrow\mathbf{R}$ *such that*

$$\omega(x,y(x))=2\left(\pi-\frac{\mathrm{sign}(x,y(x))}{1+f^2(x,y)}\right)$$

for $\forall x\in I$ *with*

$$\omega(x_0,y(x_0))=2\left(\pi-\frac{\mathrm{sign}(x,y(x))}{1+f^2(x_0,y(x_0))}\right).$$

Proof According to Theorem 8.1.3, a curve passing through the point $(x_0,y(x_0))$ in pseudo-plane $(\sum,\omega)$ if and only if $y(x_0)=y_0$ and for $\forall x\in I$,

$$\left(\pi-\frac{\omega(x,y(x))}{2}\right)\left(1+\left(\frac{dy}{dx}\right)^2\right)=\mathrm{sign}(x,y(x)).$$

Solving $\omega(x,y(x))$ from this equation, we get that

$$\omega(x,y(x))=2\left(\pi-\frac{\mathrm{sign}(x,y(x))}{1+\left(\frac{dy}{dx}\right)^2}\right)=2\left(\pi-\frac{sign(x,y(x))}{1+f^2(x,y)}\right).\qquad\square$$

Now we consider curves with an constant angle function at each of its point following.

Theorem 8.2.3 *Let* $(\sum,\omega)$ *be a pseudo-plane and* θ *a constant with* $0<\theta\leq 4\pi$.

(1) *A curve C passing through a point* (x_0,y_0) *with* $\omega(x,y)=\eta$ *for* $\forall(x,y)\in C$ *is closed without self-intersections on* $(\sum,\omega)$ *if and only if there exists a real number s such that*

$$s\eta=2(s-2)\pi.$$

(2) *A curve C passing through a point* (x_0, y_0) *with* $\omega(x, y) = \theta$ *for* $\forall(x, y) \in C$ *is a circle on* $(\sum, \omega)$ *if and only if*

$$\eta = 2\pi - \frac{2}{r},$$

where $r = \sqrt{x_0^2 + y_0^2}$, *i.e., C is a projection of a section circle passing through a point* (x_0, y_0) *on the plane XOY.*

Proof Similar to Theorem 7.3.1, we know that a curve C passing through a point (x_0, y_0) in pseudo-plane $(\sum, \omega)$ is closed if and only if

$$\int_0^s \left(\pi - \frac{\omega(s)}{2}\right) ds = 2\pi.$$

Now by assumption $\omega(x, y) = \eta$ is constant for $\forall(x, y) \in C$, we get that

$$\int_0^s \left(\pi - \frac{\omega(s)}{2}\right) ds = s\left(\pi - \frac{\eta}{2}\right).$$

Whence,

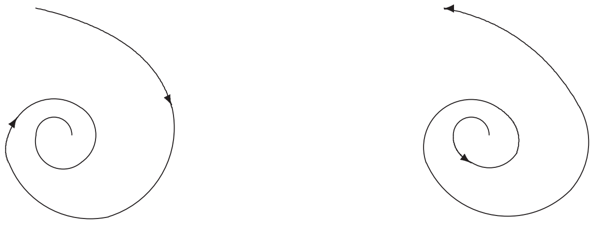
$$s(\pi - \frac{\eta}{2}) = 2\pi, \quad \text{i.e.,} \quad s\eta = 2(s - 2)\pi.$$

Now if C is a circle passing through point (x_0, y_0) with $\omega(x, y) = \theta$ for $\forall(x, y) \in C$, then by the Euclid plane geometry we know that $s = 2\pi r$, where $r = \sqrt{x_0^2 + y_0^2}$. Therefore, there must be that

$$\eta = 2\pi - \frac{2}{r}.$$

This completes the proof. □

8.2.2 Spiral Curve. Two spiral curves without self-intersections are shown in Fig.8.2.1, in where (a) is an input but (b) an output curve.



(a) (b)

Fig.8.2.1

The curve in Fig.8.2.1(a) is called an *elliptic in-spiral* and that in Fig.8.2.1(b) an *elliptic out-spiral*, correspondent to the right hand rule. In a polar coordinate system (ρ, θ), a spiral curve has equation

$$\rho = ce^{\theta t},$$

where c, t are real numbers and $c > 0$. If $t < 0$, then the curve is an in-spiral as the curve in Fig.8.2.1(a). If $t > 0$, then the curve is an out-spiral as shown in Fig.8.2.1(b).

For the case $t = 0$, we get a circle $\rho = c$ (or $x^2 + y^2 = c^2$ in the Descartes coordinate system).

Now in a pseudo-plane, we can easily find conditions for in-spiral or out-spiral curves. That is the following theorem.

Theorem 8.2.4 *Let* $(\sum, \omega)$ *be a pseudo-plane and let* η, ζ *be constants. Then an elliptic in-spiral curve* C *with* $\omega(x, y) = \eta$ *for* $\forall(x, y) \in C$ *exists in* $(\sum, \omega)$ *if and only if there exist numbers* $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ *for* $i \geq 1$ *such that*

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ *and an elliptic out-spiral curve* C *with* $\omega(x, y) = \zeta$ *for* $\forall(x, y) \in C$ *exists in* $(\sum, \omega)$ *if and only if there exist numbers* $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ *for* $i \geq 1$ *such that*

$$s_i\zeta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof Let L be an s-line like an elliptic in-spiral shown in Fig.8.2.2, in where x_1, $x_2, \cdots, x_n$ are non-Euclidean points and x_1x_6 is an auxiliary line segment.

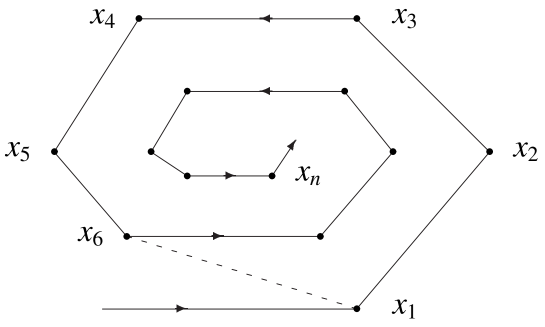


Fig.8.2.2

Then we know that

$$\sum_{i=1}^{6}(\pi - f(x_1)) < 2\pi,$$

$$\sum_{i=1}^{12}(\pi - f(x_1)) < 4\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Similarly, from any initial point O to a point P far s to O on C, the sum of lost angles at P is

$$\int_0^s \left(\pi - \frac{\eta}{2}\right) ds = \left(\pi - \frac{\eta}{2}\right) s.$$

Whence, the curve C is an elliptic in-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for ≥ 1 such that

$$\left(\pi - \frac{\eta}{2}\right) s_1 < 2\pi,$$

$$\left(\pi - \frac{\eta}{2}\right) s_2 < 4\pi,$$

$$\left(\pi - \frac{\eta}{2}\right) s_3 < 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\left(\pi - \frac{\eta}{2}\right) s_l < 2l\pi.$$

Therefore,

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Similarly, consider an s-line like an elliptic out-spiral with $x_1, x_2, \cdots, x_n$ non-Euclidean points. We can also find that C is an elliptic out-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$\left(\pi - \frac{\zeta}{2}\right) s_1 > 2\pi,$$

$$\left(\pi - \frac{\zeta}{2}\right) s_2 > 4\pi,$$

$$\left(\pi - \frac{\zeta}{2}\right) s_3 > 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\left(\pi - \frac{\zeta}{2}\right) s_l > 2l\pi.$$

Consequently,

$$s_i\eta < 2(s_i - 2i)\pi.$$

for any integer $i, i \geq 1$. □

Similar to elliptic in or out-spirals, we can also define a *hyperbolic in-spiral* or *hyperbolic out-spiral* correspondent to the left hand rule, which are mirrors of curves in Fig.8.2.1. We get the following result for a hyperbolic in or out-spiral in pseudo-plane.

Theorem 8.2.5 *Let $(\sum, \omega)$ be a pseudo-plane and let η, ζ be constants. Then a hyperbolic in-spiral curve C with $\omega(x, y) = \eta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that*

$$s_i\eta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ and a hyperbolic out-spiral curve C with $\omega(x, y) = \zeta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$s_i\zeta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof The proof is similar to that of the proof of Theorem 8.2.4. □

§8.3 STABILITY OF DIFFERENTIAL EQUATIONS

8.3.1 Singular Point. For an ordinary differential equation system

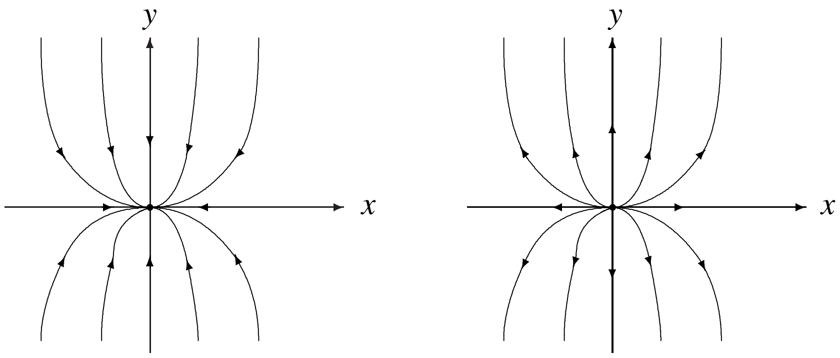
$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y), \qquad (8-1)$$

where t is a time parameter, the Euclid plane XOY with the Descartes coordinate system is called its a *phase plane* and the orbit $(x(t), y(t))$ of its a solution $x = x(t), y = y(t)$ is called an *orbit curve*. If there exists a point (x_0, y_0) on XOY such that

$$P(x_0, y_0) = Q(x_0, y_0) = 0,$$

then there is an obit curve which is only a point (x_0, y_0) on XOY. The point (x_0, y_0) is called a *singular point of* (A^*). Singular points of an ordinary differential equation are classified into four classes: *knot, saddle, focal* and *central points*. Each of these classes are introduced in the following.

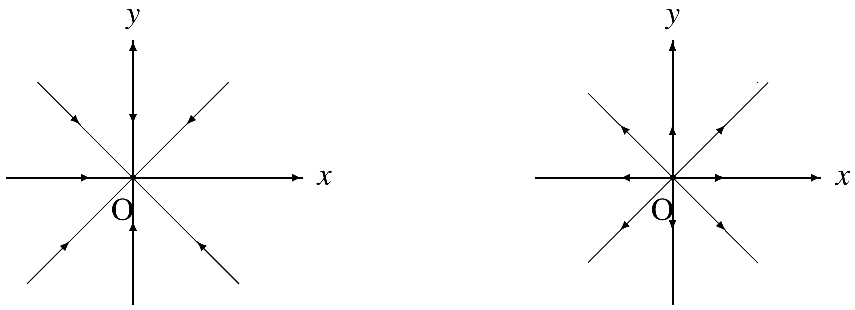
Class 1: Knot. A *knot O* of a differential equation is shown in Fig.8.3.1 where (*a*) denotes that *O* is stable but (*b*) is unstable.



(a) (b)

Fig.8.3.1

A *critical knot O* of a differential equation is shown in Fig.8.3.2 where (*a*) denotes that *O* is stable but (*b*) is unstable.



(a) (b)

Fig.8.3.2

A *degenerate knot O* of a differential equation is shown in Fig.8.3.3, where (*a*) denotes that *O* is stable but (*b*) is unstable.

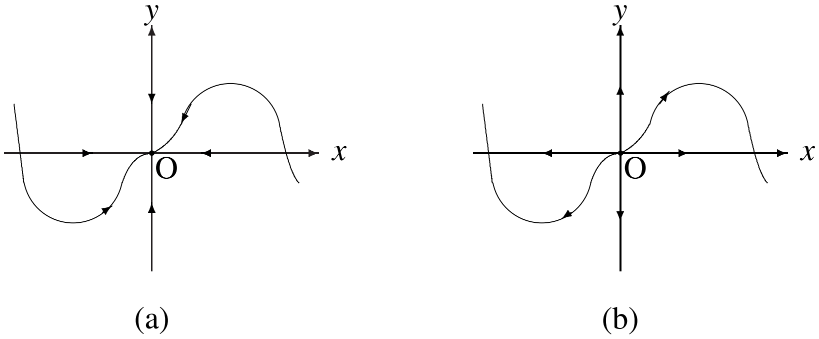


Fig.8.3.3

Class 2: Saddle Point. A *saddle point O* of a differential equation is shown in Fig.8.3.4.

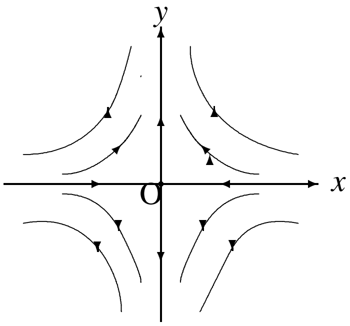


Fig.8.3.4

Class 3: Focal Point. A *focal point O* of a differential equation is shown in Fig.8.3.5, where (*a*) denotes that *O* is stable but (*b*) is unstable.

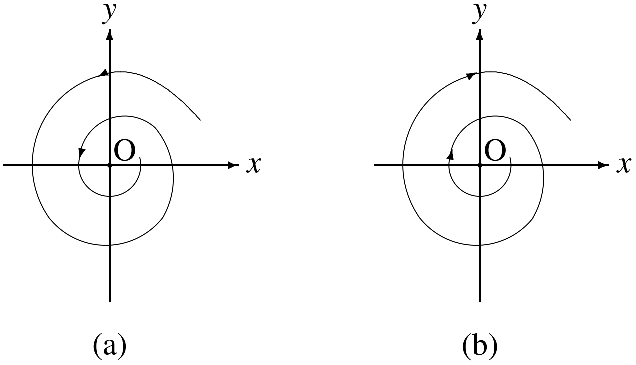


Fig.8.3.5

Class 4: Central Point. A *central point O* of a differential equation is shown in Fig.8.3.6, which is just the center of a circle.

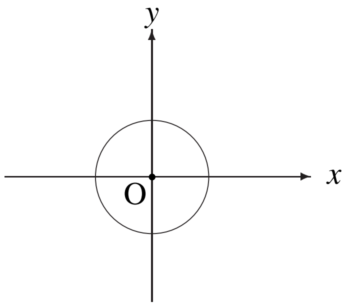


Fig.8.3.6

8.3.2 Singular Points in Pseudo-Plane. In a pseudo-plane $(\sum, \omega)$, not all kinds of singular points exist. We get a result for singular points in a pseudo-plane as in the following.

Theorem 8.3.1 *There are no saddle points and stable knots in a pseudo-plane plane* $(\sum, \omega)$.

Proof On a saddle point or a stable knot O, there are two rays to O, seeing Fig.8.3.1(a) and Fig.8.3.5 for details. Notice that if this kind of orbit curves in Fig.8.3.1(a) or Fig.8.3.5 appears, then there must be that

$$\omega(O) = 4\pi.$$

Now by Theorem 8.1.1, every point u on those two rays should be Euclidean, i.e., $\omega(u) = 2\pi$, unless the point O. But then ω is not continuous at the point O, which contradicts Definition 8.1.1. □

If an ordinary differential equation system $(8-1)$ has a closed orbit curve C but all other orbit curves are not closed in a neighborhood of C nearly enough to C and those orbits curve tend to C when $t \to +\infty$ or $t \to -\infty$, then C is called a *limiting ring of* $(8-1)$ and *stable* or *unstable* if $t \to +\infty$ or $t \to -\infty$.

Theorem 8.3.2 *For two constants* ρ_0, θ_0, $\rho_0 > 0$ *and* $\theta_0 \neq 0$, *there is a pseudo-plane* $(\sum, \omega)$ *with*

$$\omega(\rho, \theta) = 2\left(\pi - \frac{\rho_0}{\theta_0 \rho}\right)$$

or

$$\omega(\rho,\theta)=2\left(\pi+\frac{\rho_0}{\theta_0\rho}\right)$$

such that

$$\rho=\rho_0$$

is a limiting ring in $(\sum,\omega)$.

Proof Notice that for two given constants ρ_0,θ_0, $\rho_0>0$ and $\theta_0\neq 0$, the equation

$$\rho(t)=\rho_0 e^{\theta_0\theta(t)}$$

has a stable or unstable limiting ring

$$\rho=\rho_0$$

if $\theta(t)\rightarrow 0$ when $t\rightarrow+\infty$ or $t\rightarrow-\infty$. Whence, we know that

$$\theta(t)=\frac{1}{\theta_0}\ln\frac{\rho_0}{\rho(t)}.$$

Therefore,

$$\frac{d\theta}{d\rho}=\frac{\rho_0}{\theta_0\rho(t)}.$$

According to Theorem 8.1.4, we get that

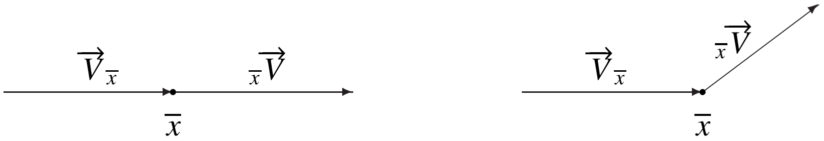
$$\omega(\rho,\theta)=2\left(\pi-sign(\rho,\theta)\frac{d\theta}{d\rho}\right),$$

for any point $(\rho,\theta)\in\sum$, i.e.,

$$\omega(\rho,\theta)=2\left(\pi-\frac{\rho_0}{\theta_0\rho}\right)\quad\text{or}\quad\omega(\rho,\theta)=2\left(\pi+\frac{\rho_0}{\theta_0\rho}\right).\qquad\square$$

§8.4 PSEUDO-EUCLIDEAN GEOMETRY

8.4.1 Pseudo-Euclidean Geometry. Let $\mathbf{R}^n=\{(x_1,x_2,\cdots,x_n)\}$ be a Euclidean space of dimensional n with a normal basis $\overline{\epsilon}_1=(1,0,\cdots,0)$, $\overline{\epsilon}_2=(0,1,\cdots,0)$, $\cdots$, $\overline{\epsilon}_n=(0,0,\cdots,1)$, $\overline{x}\in\mathbf{R}^n$ and $\overrightarrow{V}_{\overline{x}}$, ${}_{\overline{x}}\overrightarrow{V}$ two vectors with end or initial point at $\overline{x}$, respectively. A *pseudo-Euclidean space* $(\mathbf{R}^n,\mu)$ is such a Euclidean space $\mathbf{R}^n$ associated with a mapping $\mu:\overrightarrow{V}_{\overline{x}}\rightarrow{}_{\overline{x}}\overrightarrow{V}$ for $\overline{x}\in\mathbf{R}^n$, such as those shown in Fig.8.4.1,


$\overrightarrow{V}_{\overline{x}}$
$_{\overline{x}}\overrightarrow{V}$
$\overline{x}$
$\overrightarrow{V}_{\overline{x}}$
$_{\overline{x}}\overrightarrow{V}$
$\overline{x}$

(a) (b)

Fig.8.4.1

where $\overrightarrow{V}_{\overline{x}}$ and $_{\overline{x}}\overrightarrow{V}$ are in the same orientation in case (a), but not in case (b). Such points in case (a) are called *Euclidean* and in case (b) *non-Euclidean*. A pseudo-Euclidean $(\mathbf{R}^n, \mu)$ is *finite* if it only has finite non-Euclidean points, otherwise, *infinite*.

A straight line L passing through a point $(x_1^0, x_2^0, \cdots, x_n^0)$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ is defined to be a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + tX_1 \\ x_2 = x_2^0 + tX_2 \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + tX_n \end{cases}$$

for $\forall t \in \mathbf{R}$ in analytic geometry on $\mathbf{R}^n$, or equivalently, by the equation system

$$\frac{x_1 - x_1^0}{X_1} = \frac{x_2 - x_2^0}{X_2} = \cdots = \frac{x_n - x_n^0}{X_n}.$$

Therefore, we can also determine its equation system for a straight line L in a pseudo-Euclidean space $(\mathbf{R}^n, \mu)$. By definition, a straight line L passing through a Euclidean point $\overline{x}^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in \mathbf{R}^n$ with an orientation $\overrightarrow{O} = (X_1, X_2, \cdots, X_n)$ in $(\mathbf{R}^n, \mu)$ is a point set $(x_1, x_2, \cdots, x_n)$ determined by an equation system

$$\begin{cases} x_1 = x_1^0 + t(X_1 + \mu_1(\overline{x}^0)) \\ x_2 = x_2^0 + t(X_2 + \mu_2(\overline{x}^0)) \\ \cdots\cdots\cdots\cdots \\ x_n = x_n^0 + t(X_n + \mu_n(\overline{x}^0)) \end{cases}$$

for $\forall t \in \mathbf{R}$, or equivalently,

$$\frac{x_1 - x_1^0}{X_1 + \mu_1(\overline{x}^0)} = \frac{x_2 - x_2^0}{X_2 + \mu_2(\overline{x}^0)} = \cdots = \frac{x_n - x_n^0}{X_n + \mu_n(\overline{x}^0)},$$

where $\mu|_{\overrightarrow{O}}(\overline{x}^0) = (\mu_1(\overline{x}^0), \mu_2(\overline{x}^0), \cdots, \mu_n(\overline{x}^0))$. Notice that this equation system dependent on $\mu|_{\overrightarrow{O}}$, it maybe not a linear equation system.

Similarly, let $\overrightarrow{O}$ be an orientation. A point $\overline{u} \in \mathbf{R}^n$ is said to be *Euclidean* on orientation $\overrightarrow{O}$ if $\mu|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$. Otherwise, let $\mu|_{\overrightarrow{O}}(\overline{u}) = (\mu_1(\overline{u}), \mu_2(\overline{u}), \cdots, \mu_n(\overline{u}))$. The point $\overline{u}$ is *elliptic* or *hyperbolic* determined by the following inductive programming.

STEP 1. If $\mu_1(\overline{u}) < 0$, then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\mu_1(\overline{u}) > 0$;

STEP 2. If $\mu_1(\overline{u}) = \mu_2(\overline{u}) = \cdots = \mu_i(\overline{u} = 0$, but $\mu_{i+1}(\overline{u} < 0$ then $\overline{u}$ is elliptic; otherwise, hyperbolic if $\mu_{i+1}(\overline{u}) > 0$ for an integer $i, 0 \leq i \leq n-1$.

Denote these elliptic, Euclidean and hyperbolic point sets by

$$\overrightarrow{V}_{eu} = \{\, \overline{u} \in \mathbf{R}^n \mid \overline{\mathrm{u}} \text{ an Euclidean point} \,\},$$

$$\overrightarrow{V}_{el} = \{\, \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{v}} \text{ an elliptic point} \,\}.$$

$$\overrightarrow{V}_{hy} = \{\, \overline{v} \in \mathbf{R}^n \mid \overline{\mathrm{w}} \text{ a hyperbolic point} \,\}.$$

Then we get a partition

$$\mathbf{R}^n = \overrightarrow{V}_{eu} \bigcup \overrightarrow{V}_{el} \bigcup \overrightarrow{V}_{hy}$$

on points in $\mathbf{R}^n$ with $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{el} = \emptyset$, $\overrightarrow{V}_{eu} \cap \overrightarrow{V}_{hy} = \emptyset$ and $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy} = \emptyset$. Points in $\overrightarrow{V}_{el} \cap \overrightarrow{V}_{hy}$ are called *non-Euclidean points*.

Now we introduce a linear order $\prec$ on $\mathscr{O}$ by the dictionary arrangement in the following.

For $(x_1, x_2, \cdots, x_n)$ *and* $(x_1', x_2', \cdots, x_n') \in \mathscr{O}$, *if* $x_1 = x_1', x_2 = x_2', \cdots, x_l = x_l'$ *and* $x_{l+1} < x_{l+1}'$ *for any integer* $l, 0 \leq l \leq n-1$, *then define* $(x_1, x_2, \cdots, x_n) \prec (x_1', x_2', \cdots, x_n')$.

By this definition, we know that

$$\mu|_{\overrightarrow{O}}(\overline{u}) \prec \mu|_{\overrightarrow{O}}(\overline{v}) \prec \mu|_{\overrightarrow{O}}(\overline{w})$$

for $\forall \overline{u} \in \overrightarrow{V}_{el}, \overline{v} \in \overrightarrow{V}_{eu}, \overline{w} \in \overrightarrow{V}_{hy}$ and a given orientation $\overrightarrow{O}$. This fact enables us to find an interesting result following.

Theorem 8.4.1 *For any orientation* $\overrightarrow{O} \in \mathscr{O}$ *in a pseudo-Euclidean space* $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$, *if* $\overrightarrow{V}_{el} \neq \emptyset$ *and* $\overrightarrow{V}_{hy} \neq \emptyset$, *then* $\overrightarrow{V}_{eu} \neq \emptyset$.

Proof By assumption, $\overrightarrow{V}_{el} \neq \emptyset$ and $\overrightarrow{V}_{hy} \neq \emptyset$, we can choose points $\overline{u} \in \overrightarrow{V}_{el}$ and $\overline{w} \in \overrightarrow{V}_{hy}$. Notice that $\mu|_{\overrightarrow{O}} : \mathbf{R}^n \rightarrow \mathscr{O}$ is a continuous and $(\mathscr{O}, \prec)$ a linear ordered set. Applying the *generalized intermediate value theorem* on continuous mappings in topology, i.e., *Let $f : X \to Y$ be a continuous mapping with X a connected space and Y a linear ordered set in the order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.*

we know that there is a point $\overline{v} \in \mathbf{R}^n$ such that

$$\mu|_{\overrightarrow{O}}(\overline{v}) = \overline{0},$$

i.e., $\overline{v}$ is a Euclidean point by definition. □

Corollary 8.4.1 *For any orientation $\overrightarrow{O} \in \mathscr{O}$ in a pseudo-Euclidean space $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$, if $\overrightarrow{V}_{eu} = \emptyset$, then either points in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ is elliptic or hyperbolic.*

A pseudo-Euclidean space $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ is a Smarandache geometry sometimes.

Theorem 8.4.2 *A pseudo-Euclidean space $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ is a Smarandache geometry if $\overrightarrow{V}_{eu}$, $\overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$.*

Proof Notice that $\mu|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ is an axiom in $\mathbf{R}^n$, but a Smarandachely denied axiom if $\overrightarrow{V}_{eu}, \overrightarrow{V}_{el} \neq \emptyset$, or $\overrightarrow{V}_{eu}, \overrightarrow{V}_{hy} \neq \emptyset$, or $\overrightarrow{V}_{el}, \overrightarrow{V}_{hy} \neq \emptyset$ for an orientation $\overrightarrow{O}$ in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ for $\mu|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ or $\neq \overline{0}$ in the former two cases and $\mu|_{\overrightarrow{O}}(\overline{u}) \prec \overline{0}$ or $\succ \overline{0}$ both hold in the last one. Whence, we know that $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ is a Smarandache geometry by definition. □

Notice that there infinite points are on a straight line segment in $\mathbf{R}^n$. Whence, a necessary for the existence of a straight line is there exist infinite Euclidean points in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$. Furthermore, we get conditions for a curve C existing in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ following.

Theorem 8.4.2 *A curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ for an orientation $\overrightarrow{O}$ if and only if*

$$\left.\frac{df_1(t)}{dt}\right|_{\overline{u}} = \sqrt{\left(\frac{1}{\mu_1(\overline{u})}\right)^2 - 1},$$

$$\left.\frac{df_2(t)}{dt}\right|_{\overline{u}} = \sqrt{\left(\frac{1}{\mu_2(\overline{u})}\right)^2 - 1},$$

$$\cdots\cdots\cdots\cdots,$$

$$\left.\frac{df_n(t)}{dt}\right|_{\overline{u}} = \sqrt{\left(\frac{1}{\mu_n(\overline{u})}\right)^2 - 1}.$$

for $\forall\overline{u} \in C$*, where* $\mu|_{\overrightarrow{O}} = (\mu_1, \mu_2, \cdots, \mu_n)$.

Proof Let the angle between $\mu|_{\overrightarrow{O}}$ and $\overline{\epsilon}_i$ be θ_i, $1 \leq \theta_i \leq n$.

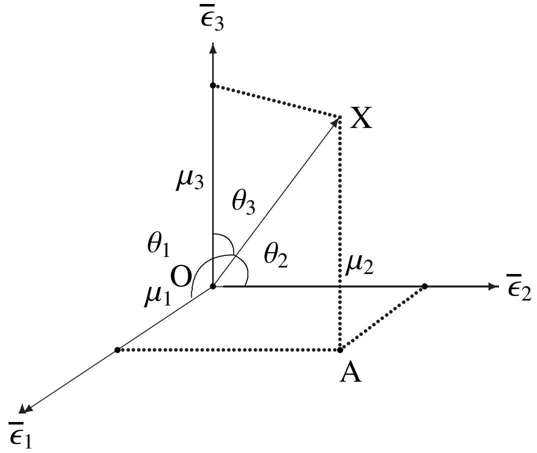


Fig.8.4.2

Then we know that

$$\cos\theta_i = \mu_i, \quad 1 \leq i \leq n.$$

By geometrical implication of differential at a point $\overline{u} \in \mathbf{R}^n$, seeing also Fig.8.4.2, we know that

$$\left.\frac{df_i(t)}{dt}\right|_{\overline{u}} = tg\theta_i = \sqrt{(\frac{1}{\mu_i(\overline{u})})^2 - 1}$$

no for $1 \leq i \leq n$. Therefore, if a curve $C = (f_1(t), f_2(t), \cdots, f_n(t))$ exists in a pseudo-Euclidean space $(\mathbf{R}^n, \mu|_{\overrightarrow{O}})$ for an orientation $\overrightarrow{O}$, then

$$\left.\frac{df_i(t)}{dt}\right|_{\overline{u}} = \sqrt{(\frac{1}{\mu_2(\overline{u})})^2 - 1}, \quad 1 \leq i \leq n$$

for $\forall\overline{u} \in C$. On the other hand, if

$$\left.\frac{df_i(t)}{dt}\right|_{\overline{v}} = \sqrt{(\frac{1}{\mu_2(\overline{v})})^2 - 1}, \quad 1 \leq i \leq n$$

hold for points $\overline{v}$ for $\forall t \in \mathbf{R}$, then all points $\overline{v}$, $t \in \mathbf{R}$ consist of a curve C in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ for the orientation $\overrightarrow{O}$. □

Corollary 8.4.2 *A straight line* L *exists in* $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ *if and only if* $\mu|_{\overrightarrow{O}}(\overline{u}) = \overline{0}$ *for* $\forall\overline{u} \in L$ *and* $\forall\overrightarrow{O} \in \mathscr{O}$.

8.4.2 Rotation Matrix. Notice that a vector $\overrightarrow{V}$ can be uniquely determined by the basis of $\mathbf{R}^n$. For $\overline{x} \in \mathbf{R}^n$, there are infinite orthogonal frames at point $\overline{x}$. Denoted by $\mathcal{O}_{\overline{x}}$ the set of all normal bases at point $\overline{x}$. Then a *pseudo-Euclidean space* $(\mathbf{R}, \mu)$ is nothing but a Euclidean space $\mathbf{R}^n$ associated with a linear mapping $\mu : \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\} \to \{\overline{\epsilon}'_1, \overline{\epsilon}'_2, \cdots, \overline{\epsilon}'_n\} \in \mathcal{O}_{\overline{x}}$ such that $\mu(\overline{\epsilon}_1) = \overline{\epsilon}'_1$, $\mu(\overline{\epsilon}_2) = \overline{\epsilon}'_2$, $\cdots$, $\mu(\overline{\epsilon}_n) = \overline{\epsilon}'_n$ at point $\overline{x} \in \mathbf{R}^n$. Thus if $\overrightarrow{V}_{\overline{x}} = c_1\overline{\epsilon}_1 + c_2\overline{\epsilon}_2 + \cdots + c_n\overline{\epsilon}_n$, then $\mu\left({}_{\overline{x}}\overrightarrow{V}\right) = c_1\mu(\overline{\epsilon}_1) + c_2\mu(\overline{\epsilon}_2) + \cdots + c_n\mu(\overline{\epsilon}_n) = c_1\overline{\epsilon}'_1 + c_2\overline{\epsilon}'_2 + \cdots + c_n\overline{\epsilon}'_n$.

Without loss of generality, assume that

$$\begin{aligned}
\mu(\overline{\epsilon}_1) &= x_{11}\overline{\epsilon}_1 + x_{12}\overline{\epsilon}_2 + \cdots + x_{1n}\overline{\epsilon}_n,\\
\mu(\overline{\epsilon}_2) &= x_{21}\overline{\epsilon}_1 + x_{22}\overline{\epsilon}_2 + \cdots + x_{2n}\overline{\epsilon}_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\mu(\overline{\epsilon}_n) &= x_{n1}\overline{\epsilon}_1 + x_{n2}\overline{\epsilon}_2 + \cdots + x_{nn}\overline{\epsilon}_n.
\end{aligned}$$

Then we find that

$$\begin{aligned}
\mu\left({}_{\overline{x}}\overrightarrow{V}\right) &= (c_1, c_2, \cdots, c_n)(\mu(\overline{\epsilon}_1), \mu(\overline{\epsilon}_2), \cdots, \mu(\overline{\epsilon}_n))^t\\
&= (c_1, c_2, \cdots, c_n)\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix}(\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n)^t.
\end{aligned}$$

Denoted by

$$[\overline{x}] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{pmatrix} = \begin{pmatrix} \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_1), \overline{\epsilon}_n\rangle\\ \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_2), \overline{\epsilon}_n\rangle\\ \cdots & \cdots & \cdots & \cdots\\ \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_1\rangle & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_2\rangle & \cdots & \langle\mu(\overline{\epsilon}_n), \overline{\epsilon}_n\rangle\end{pmatrix},$$

called the *rotation matrix* of $\overline{x}$ in $(\mathbf{R}^n, \mu)$. Then $\mu : \overrightarrow{V}_{\overline{x}} \to {}_{\overline{x}}\overrightarrow{V}$ is determined by $\mu(\overline{x}) = [\overline{x}]$ for $\overline{x} \in \mathbf{R}^n$. Furthermore, such an rotation matrix $[\overline{x}]$ is orthogonal for points $\overline{x} \in \mathbf{R}^n$ by definition, i.e., $[\overline{x}]\,[\overline{x}]^t = I_{n\times n}$. Particularly, if $\overline{x}$ is Euclidean, then such an orientation matrix is nothing but $\mu(\overline{x}) = I_{n\times n}$. Summing up all these discussions, we know the following result.

Theorem 8.4.3 *If $(\mathbf{R}^n, \mu)$ is a pseudo-Euclidean space, then $\mu(\overline{x}) = [\overline{x}]$ is an $n \times n$ orthogonal matrix for $\forall\ \overline{x} \in \mathbf{R}^n$.*

8.4.3 Finitely Pseudo-Euclidean Geometry. Let $n \geq 2$ be an integer. We can characterize finitely pseudo-Euclidean geometry by that of embedded graph in $\mathbf{R}^n$. As we known, an embedded graph G on $\mathbf{R}^n$ is a $1-1$ mapping $\tau : G \to \mathbf{R}^n$ such that for $\forall e, e' \in E(G)$, $\tau(e)$ has no self-intersection and $\tau(e)$, $\tau(e')$ maybe only intersect at their end points. Such an embedded graph G in $\mathbf{R}^n$ is denoted by $G_{\mathbf{R}^n}$.

Likewise that the case of $(\mathbf{R}^2, \mu)$, the *curvature* $R(L)$ of an s-line L passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$ for $m \geq 0$ in $(\mathbf{R}^n, \mu)$ to be a matrix determined by

$$R(L) = \prod_{i=1}^{m} \mu(\overline{x}_i)$$

and *Euclidean* if $R(L) = I_{n\times n}$, otherwise, *non-Euclidean*. obviously, a point in a Euclidean space $\mathbf{R}^n$ is indeed Euclidean by this definition. Furthermore, we immediately get the following result for Euclidean s-lines in $(\mathbf{R}^n, \mu)$.

Theorem 8.4.4 *Let $(\mathbf{R}^n, \mu)$ be a pseudo-Euclidean space and L an s-line in $(\mathbf{R}^n, \mu)$ passing through non-Euclidean points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m \in \mathbf{R}^n$. Then L is closed if and only if L is Euclidean.*

Proof If L is a closed s-line, then L is consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$. By definition,

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\,\mu(\overline{x}_i)$$

for integers $1 \leq i \leq m$, where $i + 1 \equiv (\mathrm{mod} m)$. Consequently,

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2} \prod_{i=1}^{m} \mu(\overline{x}_i).$$

Thus $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, i.e., L is Euclidean.

Conversely, let L be Euclidean, i.e., $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$. By definition, we know that

$$\frac{\overrightarrow{\overline{x}_{i+1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|} = \frac{\overrightarrow{\overline{x}_{i-1}\overline{x}_i}}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\,\mu(\overline{x}_i), \quad \text{i.e.,} \quad \overrightarrow{\overline{x}_{i+1}\overline{x}_i} = \frac{\left|\overrightarrow{\overline{x}_{i+1}\overline{x}_i}\right|}{\left|\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\right|}\overrightarrow{\overline{x}_{i-1}\overline{x}_i}\,\mu(\overline{x}_i)$$

for integers $1 \leq i \leq m$, where $i + 1 \equiv (\mathrm{mod} m)$. Whence, if $\prod\limits_{i=1}^{m} \mu(\overline{x}_i) = I_{n\times n}$, then there must be

$$\overrightarrow{\overline{x}_1\overline{x}_2} = \overrightarrow{\overline{x}_1\overline{x}_2}\prod_{i=1}^{m}\mu(\overline{x}_i).$$

Thus L consisted of vectors $\overrightarrow{\overline{x}_1\overline{x}_2}$, $\overrightarrow{\overline{x}_2\overline{x}_3}$, $\cdots$, $\overrightarrow{\overline{x}_n\overline{x}_1}$ is a closed s-line in $(\mathbf{R}^n,\mu)$. $\square$

Similarly, an embedded graph $G_{\mathbf{R}^n}$ is called *Smarandachely* if there exists a pseudo-Euclidean space $(\mathbf{R}^n,\mu)$ with a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ such that all of its vertices are non-Euclidean points in $(\mathbf{R}^n,\mu)$. It should be noted that these vertices of valency 1 is not important for Smarandachely embedded graphs. We get a result on embedded 2-connected graphs similar to that of Theorem 6.4.2 following.

Theorem 8.4.5 *An embedded* 2*-connected graph* $G_{\mathbf{R}^n}$ *is Smarandachely if and only if there is a mapping* $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ *and a directed circuit-decomposition*

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

such that these matrix equations

$$\prod_{\overline{x}\in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

are solvable.

Proof By definition, if $G_{\mathbf{R}^n}$ is Smarandachely, then there exists a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ such that all vertices of $G_{\mathbf{R}^n}$ are non-Euclidean in $(\mathbf{R}^n,\mu)$. Notice there are only two orientations on an edge in $E(G_{\mathbf{R}^n})$. Traveling on $G_{\mathbf{R}^n}$ beginning from any edge with one orientation, we get a closed s-line $\overrightarrow{C}$, i.e., a directed circuit. After we traveled all edges in $G_{\mathbf{R}^n}$ with the possible orientations, we get a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

with an s-line $\overrightarrow{C}_i$ for integers $1 \leq i \leq s$. Applying Theorem 9.4.6, we get

$$\prod_{\overline{x}\in V(\overrightarrow{C}_i)} \mu(\overline{x}) = I_{n\times n} \quad 1 \leq i \leq s.$$

Thus these equations

$$\prod_{\overline{x}\in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

have solutions $X_{\overline{x}} = \mu(\overline{x})$ for $\overline{x} \in V\left(\overrightarrow{C}_i\right)$.

Conversely, if these is a directed circuit-decomposition

$$E_{\frac{1}{2}} = \bigoplus_{i=1}^{s} E\left(\overrightarrow{C}_i\right)$$

such that these matrix equations

$$\prod_{\overline{x} \in V(\overrightarrow{C}_i)} X_{\overline{x}} = I_{n\times n} \quad 1 \leq i \leq s$$

are solvable, let $X_{\overline{x}} = A_{\overline{x}}$ be such a solution for $\overline{x} \in V\left(\overrightarrow{C}_i\right),\ 1 \leq i \leq s$. Define a mapping $\mu : \overline{x} \in \mathbf{R}^n \to [\overline{x}]$ on $\mathbf{R}^n$ by

$$\mu(\overline{x}) = \begin{cases} A_{\overline{x}} & \text{if } \overline{x} \in V(G_{\mathbf{R}^n}), \\ I_{n\times n} & \text{if } \overline{x} \notin V(G_{\mathbf{R}^n}). \end{cases}$$

Thus we get a Smarandachely embedded graph $G_{\mathbf{R}^n}$ in the pseudo-Euclidean space $(\mathbf{R}^n, \mu)$ by Theorem 8.4.4. □

8.4.4 Metric Pseudo-Geometry. We can further generalize Definition 8.1.1 and get Smarandache geometry on metric spaces following.

Definition 8.4.1 *Let U and W be two metric spaces with metric ρ, $W \subseteq U$. For $\forall u \in U$, if there is a continuous mapping $\omega : u \to \omega(u)$, where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in W$, $\rho(u - v) < \delta$ such that $\rho(\omega(u) - \omega(v)) < \epsilon$, then U is called a metric pseudo-space if $U = W$ or a bounded metric pseudo-space if there is a number $N > 0$ such that $\forall w \in W$, $\rho(w) \leq N$, denoted by (U, ω) or (U^-, ω), respectively.*

By choice different metric spaces U and W in this definition, we can get various metric pseudo-spaces. Particularly, for $n = 1$, we can also explain $\omega(u)$ being an angle function with $0 < \omega(u) \leq 4\pi$, i.e.,

$$\omega(u) = \begin{cases} \omega(u)(mod 4\pi), & \text{if u} \in \text{W}, \\ 2\pi, & \text{if u} \in \text{U} \setminus \text{W}. \end{cases}$$

The following result convinces us that there are Smarandache geometries in metric pseudo-spaces.

Theorem 8.4.5 *For any integer $n \geq 1$, there are infinite Smarandache geometries in metric pseudo-spaces or bounded metric pseudo-spaces M.*

Proof Let Δ and Λ be subset of $\mathbf{R}^n$ or $\mathbf{C}^n$ with $\Delta \bigcap \Lambda = \emptyset$, W a bounded subspace of M and let $W_1, W_2 \subset W$ with $W_1 \bigcap W_2 = \emptyset$. Since M is a metric space and $W_1 \bigcap W_2 = \emptyset$, $\Delta \bigcap \Lambda = \emptyset$, we can always define a continuous mapping $\omega : u \to \omega(u)$ on W such that

$$\omega(w_1) \in \Delta \text{ for } w_1 \in W_1; \quad \omega(w_2) \in \Lambda \text{ for } w_2 \in W_2.$$

Therefore, the statement $\omega(u) \in \Delta$ *for any point* $u \in M$ is Smarandachely denied by the definition of ω, i.e., $\omega(w_1) \in \Delta$ for $w_1 \in W_1$, $\omega(w_2) \in \Lambda$ for $w_2 \in W_2$ and $\omega(w)$ for $w \in M \setminus (W_1 \bigcup W_2)$ or $\omega(u)$ for $u \in (M \setminus W)$ can be defined as we wish since $W_1 \bigcap W_2 = \emptyset$ and $W \setminus (W_1 \bigcup W_2) \neq \emptyset$, $M \setminus W \neq \emptyset$. By definition, we get a Smarandache geometry (M, ω) with or without boundary. □

§8.5 SMOOTH PSEUDO-MANIFOLDS

8.5.1 Differential Manifold. A *differential n-manifold* $(M^n, \mathscr{A})$ is an n-manifold M^n, where $M^n = \bigcup\limits_{i \in I} U_i$ endowed with a C^r-differential structure $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha) \ \text{ are } \ C^r;$$

(3) $\mathscr{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathscr{A}$, then $(U, \varphi) \in \mathscr{A}$.

An n-manifold is called to be *smooth* if it is endowed with a C^∞-differential structure. It has been known that the base of a tangent space $T_p M^n$ of differential n-manifold $(M^n, \mathscr{A})$ consisting of $\dfrac{\partial}{\partial x^i}$, $1 \leq i \leq n$ for $\forall p \in (M^n, \mathscr{A})$.

8.5.2 Pseudo-Manifold. An *n-dimensional pseudo-manifold* $(M^n, \mathscr{A}^\mu)$ is a Hausdorff space such that each points p has an open neighborhood U_p homomorphic to a pseudo-Euclidean space $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$, where $\mathscr{A} = \{(U_p, \varphi_p^\mu) | p \in M^n\}$ is its atlas with a homomorphism $\varphi_p^\mu : U_p \to \left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$ and a chart (U_p, φ_p^μ).

Theorem 8.5.1 *For a point $p \in (M^n, \mathcal{A}^\mu)$ with a local chart (U_p, φ_p^μ), $\varphi_p^\mu = \varphi_p$ if and only if $\mu|_{\overrightarrow{O}}(p) = \overline{0}$.*

Proof For $\forall p \in (M^n, \mathcal{A}^\mu)$, if $\varphi_p^\mu(p) = \varphi_p(p)$, then $\mu(\varphi_p(p)) = \varphi_p(p)$. By the definition of pseudo-Euclidean space $(\mathbf{R}^n, \mu|_{\overrightarrow{O}})$, this can only happens while $\mu(p) = \overline{0}$. □

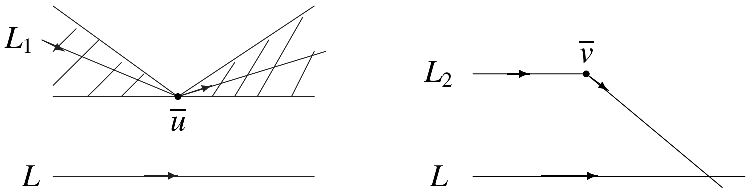
A point $p \in (M^n, \mathcal{A}^\mu)$ is *elliptic, Euclidean* or *hyperbolic* if $\mu(\varphi_p(p)) \in (\mathbf{R}^n, \mu|_{\overrightarrow{O}})$ is *elliptic, Euclidean* or *hyperbolic*, respectively. These elliptic and hyperbolic points also called *non-Euclidean points*. We get a consequence by Theorem 8.5.1.

Corollary 8.5.1 *Let $(M^n, \mathcal{A}^\mu)$ be a pseudo-manifold. Then $\varphi_p^\mu = \varphi_p$ if and only if every point in M^n is Euclidean.*

Theorem 8.5.2 *Let $(M^n, \mathcal{A}^\mu)$ be an n-dimensional pseudo-manifold, $p \in M^n$. If there are Euclidean and non-Euclidean points simultaneously or two elliptic or hyperbolic points on an orientation $\overrightarrow{O}$ in (U_p, φ_p), then $(M^n, \mathcal{A}^\mu)$ is a paradoxist n-manifold.*

Proof Notice that two lines L_1, L_2 are said *locally parallel* in a neighborhood (U_p, φ_p^μ) of a point $p \in (M^n, \mathcal{A}^\mu)$ if $\varphi_p^\mu(L_1)$ and $\varphi_p^\mu(L_2)$ are parallel in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$. If these conditions hold for $(M^n, \mathcal{A}^\mu)$, the axiom that *there is exactly one line passing through a point locally parallel a given line* is Smarandachely denied since it behaves in at least two different ways, i.e., *one parallel, none parallel*, or *one parallel, infinite parallels*, or *none parallel, infinite parallels*, which are verified in the following.

If there are Euclidean and non-Euclidean points in (U_p, φ_p^μ) simultaneously, not loss of generality, let u be Euclidean but v non-Euclidean, $\varphi_p^\mu(v) = (\mu_1, \mu_2, \cdots, \mu_n)$ with $\mu_1 < 0$.



(a) (b)

Fig.8.5.1

Let L be a line parallel the axis $\overline{\epsilon}_1$ in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$. Then there is only one line L_u locally parallel to $(\varphi_p^\mu)^{-1}(L)$ passing through the point u since there is only one line $\varphi_p^\mu(L_u)$ parallel to L in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$. However, if $\mu_1 > 0$, then there are infinite many lines passing through u locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) because there are infinite many lines parallel L in

$\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$, such as those shown in Fig.8.5.1(a), in where each line passing through the point $\overline{u} = \varphi_p^{\mu}(u)$ from the shade field is parallel to L. But if $\mu_1 > 0$, then there are no lines locally parallel to $(\varphi_p^{\mu})^{-1}(L)$ in (U_p, φ_p^{μ}) since there are no lines passing through the point $\overline{v} = \varphi_p^{\mu}(v)$ parallel to L in $\left(\mathbf{R}^n, \mu|_{\overrightarrow{O}}\right)$, such as those shown in Fig.8.5.1(b).

If there are two elliptic points u, v along a direction $\overrightarrow{O}$, consider the plane $\sum$ determined by $\varphi_p^{\omega}(u), \varphi_p^{\omega}(v)$ with $\overrightarrow{O}$ in $\left(\mathbf{R}^n, \omega|_{\overrightarrow{O}}\right)$. Let L be a line intersecting with the line $\varphi_p^{\omega}(u)\varphi_p^{\omega}(v)$ in $\sum$. Then there are infinite lines passing through u locally parallel to $(\varphi_p^{\omega})^{-1}(L)$ but none line passing through v locally parallel to $\varphi_p^{-1}(L)$ in (U_p, φ_p) because there are infinite many lines or none lines passing through $\overline{u} = \varphi_p^{\omega}(u)$ or $\overline{v} = \varphi_p^{\omega}(v)$ parallel to L in $\left(\mathbf{R}^n, \omega|_{\overrightarrow{O}}\right)$, such as those shown in Fig.8.5.2.

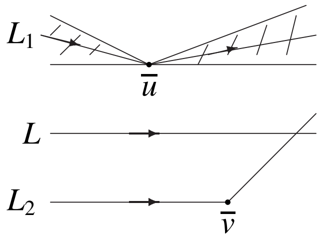


Fig.8.5.2

For the case of hyperbolic points, we can similarly get the conclusion. Since there exists a Smarandachely denied axiom to the fifth Euclid's axiom in $(M^n, \mathscr{A}^{\omega})$, it is indeed a paradoxist manifold. □

If $M = \mathbf{R}^n$, we get consequences for pseudo-Euclidean spaces $\left(\mathbf{R}^n, \omega|_{\overrightarrow{O}}\right)$ following.

Corollary 8.5.2 *For an integer $n \geq 2$, if there are Euclidean and non-Euclidean points simultaneously or two elliptic or hyperbolic points in an orientation $\overrightarrow{O}$ in $\left(\mathbf{R}^n, \omega|_{\overrightarrow{O}}\right)$, then $\left(\mathbf{R}^n, \omega|_{\overrightarrow{O}}\right)$ is a paradoxist n-manifold.*

Corollary 8.5.3 *If there are points $\overline{p}, \overline{q} \in \left(\mathbf{R}^3, \omega|_{\overrightarrow{O}}\right)$ such that $\omega|_{\overrightarrow{O}}(\overline{p}) \neq (0,0,0)$ but $\omega|_{\overrightarrow{O}}(\overline{q}) = (0,0,0)$ or $\overline{p}, \overline{q}$ are simultaneously elliptic or hyperbolic in an orientation $\overrightarrow{O}$ in $\left(\mathbf{R}^3, \omega|_{\overrightarrow{O}}\right)$, then $\left(\mathbf{R}^3, \omega|_{\overrightarrow{O}}\right)$ is a paradoxist n-manifold.*

8.5.3 Differential Pseudo-Manifold. For an integer $r \geq 1$, a C^r-differential pseudo-manifold $(M^n, \mathscr{A}^{\omega})$ is a pseudo-manifold $(M^n, \mathscr{A}^{\omega})$ endowed with a C^r-differentiable structure $\mathscr{A}$ and $\omega|_{\overrightarrow{O}}$ for an orientation $\overrightarrow{O}$. A C^{∞}-differential pseudo-manifold $(M^n, \mathscr{A}^{\omega})$ is also said to be a *smooth pseudo-manifold*. For such pseudo-manifolds, we know their differentiable conditions following.

Theorem 8.5.3 *A pseudo-Manifold* $(M^n, \mathcal{A}^\omega)$ *is a* C^r*-differential pseudo-manifold with an orientation* $\overrightarrow{O}$ *for an integer* $r \geq 1$ *if conditions following hold.*

(1) *There is a* C^r*-differential structure* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ *on* M^n*;*

(2) $\omega|_{\overrightarrow{O}}$ *is* C^r*;*

(3) *There are Euclidean and non-Euclidean points simultaneously or two elliptic or hyperbolic points on the orientation* $\overrightarrow{O}$ *in* (U_p, φ_p) *for a point* $p \in M^n$.

Proof The condition (1) implies that $(M^n, \mathcal{A})$ is a C^r-differential n-manifold and conditions (2) and (3) ensure $(M^n, \mathcal{A}^\omega)$ is a differential pseudo-manifold by definitions and Theorem 8.5.2. □

§8.6 RESEARCH PROBLEMS

Definition 8.4.1 is a general way for introducing pseudo-geometry on metric spaces. However, even for Euclidean plane $\sum$, there are many problems not solved yet. We list some of them on Euclidean spaces $\mathbf{R}^m$ and m-manifolds for $m \geq 2$ following.

8.6.1 Let C be a closed curve in Euclid plane $\sum$ without self-intersection. Then the curve C divides $\sum$ into two domains. One of them is finite, denoted by D_{fin}. We call C the boundary of D_{fin}. Now let $U = \sum$ and $W = D_{fin}$ in Definition 8.4.1 with $n = 1$. For example, choose C be a 6-polygon such as those shown in Fig.8.6.1.

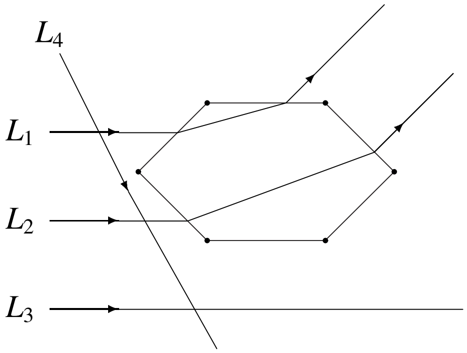


Fig.8.6.1

Then we get a geometry $(\sum^-, \omega)$ partially Euclidean, and partially non-Euclidean. Then there are open problems following.

Problem 8.6.1 *Find conditions for parallel bundles on* $(\sum^-, \omega)$.

Problem 8.6.2 *Find conditions for existing an algebraic curve* $F(x, y) = 0$ *on* $(\sum^-, \omega)$.

Problem 8.6.3 *Find conditions for existing an integer curve* C *on* $(\sum^-, \omega)$.

8.6.2 For any integer $m, m \geq 3$ and a point $\overline{u} \in \mathbf{R}^m$. Choose $U = W = \mathbf{R}^m$ in Definition 8.4.1 for $n = 1$ and $\omega(\overline{u})$ an angle function. Then we get a pseudo-space geometry $(\mathbf{R}^m, \omega)$.

Problem 8.6.4 *Find conditions for existing an algebraic surface* $F(x_1, x_2, \cdots, x_m) = 0$ *in* $(\mathbf{R}^m, \omega)$*, particularly, for an algebraic surface* $F(x_1, x_2, x_3) = 0$ *existing in* $(\mathbf{R}^3, \omega)$.

Problem 8.6.5 *Find conditions for existing an integer surface in* $(\mathbf{R}^m, \omega)$.

If we take $U = \mathbf{R}^m$ and W a bounded convex point set of $\mathbf{R}^m$ in Definition 8.4.1. Then we get a bounded pseudo-space $(\mathbf{R}^{m-}, \omega)$, which is also partially Euclidean, and partially non-Euclidean. A natural problem on $(\mathbf{R}^{m-}, \omega)$ is the following.

Problem 8.6.6 *For a bounded pseudo-space* $(\mathbf{R}^{m-}, \omega)$*, solve Problems* 8.6.4 *and* 8.6.5.

8.6.3 For a locally orientable surface S and $\forall u \in S$, choose $U = W = S$ in Definition 8.4.1 for $n = 1$ and $\omega(u)$ an angle function. Then we get a pseudo-surface geometry (S, ω).

Problem 8.6.7 *Characterize curves on a surface* S *by choice angle function* ω*. Whether can we classify automorphisms on* S *by applying pseudo-surface geometry* (S, ω)*?*

Notice that Thurston [Thu1] had classified automorphisms of surface S, $\chi(S) \leq 0$ into three classes: *reducible, periodic* or *pseudo-Anosov*. If we take $U = S$ and W a bounded simply connected domain on S in Definition 8.4.1, we get a bounded pseudo-surface (S^-, ω).

Problem 8.6.8 *For a bounded pseudo-surface* (S^-, ω)*, solve Problem* 8.6.7.

8.6.4 A *Minkowski norm* on manifold M^m is a function $F : M^m \to [0, +\infty)$ such that

(1) F is smooth on $M^m \setminus \{0\}$;

(2) F is 1-homogeneous, i.e., $F(\lambda\overline{u}) = \lambda F(\overline{u})$ for $\overline{u} \in M^m$ and $\lambda > 0$;

(3) for $\forall y \in M^m \setminus \{0\}$, the symmetric bilinear form $g_y : M^m \times M^m \to R$ with

$$g_y(\overline{u}, \overline{v}) = \frac{1}{2}\frac{\partial^2 F^2(y + s\overline{u} + t\overline{v})}{\partial s \partial t}\bigg|_{t=s=0}$$

is positive definite and a *Finsler manifold* is such a manifold M^m associated with a function $F : TM^m \to [0, +\infty)$ that

(1) F is smooth on $TM^m \setminus \{0\} = \bigcup\{T_{\overline{x}}M^m \setminus \{0\} : \overline{x} \in M^m\}$;

(2) $F|_{T_{\overline{x}}M^m} \to [0, +\infty)$ is a Minkowski norm for $\forall \overline{x} \in M^m$.

As a special case of pseudo-manifold geometry, equip a pseudo-manifold (M^m, ω) with a Minkowski norm and choose $\omega(\overline{x}) = F(\overline{x})$ for $\overline{x} \in M^m$, then (M^m, ω) is a *Finsler manifold*, particularly, if $\omega(\overline{x}) = g_{\overline{x}}(y, y) = F^2(x, y)$, then (M^m, ω) is a *Riemann manifold*. Thereby, we conclude that *the Smarandache manifolds, particularly, pseudo-manifolds include Finsler manifolds as a subset*. Open problems on pseudo-manifold geometry are listed in the following.

Problem 8.6.9 *Characterize the pseudo-manifold geometry* (M^m, ω) *without boundary and apply it to classical mathematics and mechanics.*

Similarly, if we take $U = M^m$ and W a bounded submanifold of M^m in Definition 8.4.1, we get a bounded pseudo-manifold (M^{m-}, ω).

Problem 8.6.10 *Characterize the pseudo-manifold geometry* (M^{m-}, ω) *with boundary and apply it to classical mathematics and mechanics, particularly, to hamiltonian mechanics.*

CHAPTER 9.

Spacial Combinatorics

Are all things in the WORLD out of order or in order? Different notion answers this question differently. There is well-known Chinese ancient book, namely *TAO TEH KING* written by *LAO ZI* claims that *the Tao gives birth to One; One gives birth to Two; Two gives birth to Three; Three gives birth to all things* and *all things that we can acknowledge is determined by our eyes, or ears, or nose, or tongue, or body or passions, i.e., these six organs*, which implies that all things in the WORLD is in order with patterns. Thus human beings can understand the WORLD by finding such patterns. This notion enables us to consider multi-spaces underlying combinatorial structures, i.e., *spacial combinatorics* and find their behaviors to imitate the WORLD. For this objective, we introduce the inherited combinatorial structures of Smarandache multi-spaces, such as those of multi-sets, multi-groups, multi-rings and vector multi-spaces in Section 9.1 and discuss combinatorial Euclidean spaces and combinatorial manifolds with characteristics in Sections 9.2 and 9.3. Section 9.4 concentrates on the combination of topological with those of algebraic structures, i.e., topological groups, a kind of multi-spaces known in classical mathematics and topological multi-groups. For multi-metric spaces underlying graphs, we get interesting results, particularly, a generalization of Banach's fixed point theorem in Section 9.5. All of these are an application of the combinatorial principle to Smarandace multi-spaces, i.e., Conjecture 4.5.1 (CC Conjecture) for advancing the 21st mathematical sciences presented by the author in 2005.

§9.1 COMBINATORIAL SPACES

9.1.1 Inherited Graph in Multi-Space. Let $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ be a Smarandache multi-space consisting of m spaces $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ for an integer $n \geq 2$, different two by two with

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad \text{and} \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Its underlying graph is an edge-labeled graph defined following.

Definition 9.1.1 *Let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multi-space with* $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{m} \Sigma_i$ *and* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$. *Its underlying graph* $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ *is defined by*

$$V\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right) = \{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$
$$E\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right) = \{\ (\Sigma_i,\Sigma_j)\ |\ \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i,j \leq m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i,\Sigma_j) \in E\left(G\left[\widetilde{S},\widetilde{R}\right]\right) \rightarrow l^E(\Sigma_i,\Sigma_j) = \varpi\left(\Sigma_i \bigcap \Sigma_j\right),$$

where ϖ *is a characteristic on* $\Sigma_i \bigcap \Sigma_j$ *such that* $\Sigma_i \bigcap \Sigma_j$ *is isomorphic to* $\Sigma_k \bigcap \Sigma_l$ *if and only if* $\varpi(\Sigma_i \bigcap \Sigma_j) = \varpi(\Sigma_k \bigcap \Sigma_l)$ *for integers* $1 \leq i,j,k,l \leq m$.

For understanding this inherited graph $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ of multi-space $\left(\widetilde{\Sigma};\widetilde{R}\right)$, we consider a simple case, i.e., all spaces $(\Sigma_i;\mathcal{R}_i)$ is nothing but a finite set S_i for integers $1 \leq i \leq m$. Such a multi-space $\widetilde{S}$ is called a *multi-set*. Choose the characteristic ϖ on $S_i \bigcap S_j$ to be the set $S_i \bigcap S_j$. Then we get an edge-labeled graph $G\left[\widetilde{S}\right]$. For example, let $S_1 = \{a,b,c\}$, $S_2 = \{c,d,e\}$, $S_3 = \{a,c,e\}$ and $S_4 = \{d,e,f\}$. Then the multi-set $\widetilde{S} = \bigcup\limits_{i=1}^{4} S_i = \{a,b,c,d,e,f\}$ with its edge-labeled graph $G\left[\widetilde{S}\right]$ shown in Fig.9.1.1.

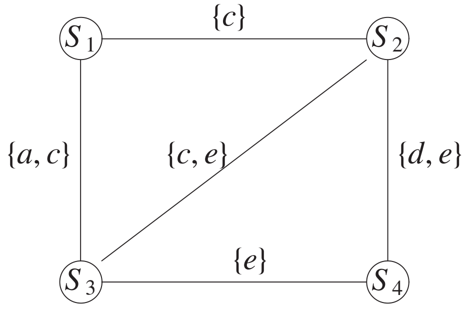


Fig.9.1.1

Theorem 9.1.1 *Let $\widetilde{S}$ be a multi-set with $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$ and $i_j \in \{1, 2, \cdots, m\}$ for integers $1 \leq s \leq m$. Then,*

$$\left|\widetilde{S}\right| \geq |\bigcup_{j=1}^{s} S_{i_j}| + m - s - 1.$$

Particularly, $\left|\widetilde{S}\right| \geq |S_i| + m - 1$ for any integer $i,\ 1 \leq i \leq m$.

Proof Notice that sets $S_i,\ 1 \leq i \leq m$ are different two by two. thus $|S_i - S_j| \geq 1$ for integers $1 \leq i, j \leq m$. Whence,

$$\left|\widetilde{S}\right| \geq \left|\bigcup_{j=1}^{s} S_{i_j}\right| + m - s - 1.$$

Particularly, let $s = 1$. We get that $|\widetilde{S}| \geq |S_i| + m - 1$ for any integer $i,\ 1 \leq i \leq m$. $\square$

Let $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$ be an n-set. It is easily to know that there are $\begin{pmatrix} n \\ m \end{pmatrix} 2^{n-m}$ sets $S_1, S_2, \cdots, S_m$, different two by two such that their union is the multi-set $\widetilde{S}$. Whence, there are

$$\sum_{2 \leq m \leq n} \begin{pmatrix} n \\ m \end{pmatrix} 2^{n-m}$$

m such sets $S_1, S_2, \cdots, S_m$ consisting the multi-set $\widetilde{S}$. By Definition 9.1.1, we can classify Smarandache multi-spaces combinatorially by introducing the following conception.

Definition 9.1.2 *Two Smarandache multi-spaces $\left(\widetilde{\Sigma}_1; \widetilde{\mathcal{R}}_1\right)$ and $\left(\widetilde{\Sigma}_2; \widetilde{\mathcal{R}}_2\right)$ are combinatorially equivalent if there is a bijection $\varphi : G\left[\widetilde{\Sigma}_1; \widetilde{\mathcal{R}}_1\right] \to G\left[\widetilde{\Sigma}_2; \widetilde{\mathcal{R}}_2\right]$ such that*

(1) *φ is an isomorphism of graph;*

(2) *If $\varphi : \Sigma_1 \in V\left(G\left[\widetilde{\Sigma}_1; \widetilde{\mathcal{R}}_1\right]\right) \to \Sigma_2 \in G\left[\widetilde{\Sigma}_2; \widetilde{\mathcal{R}}_2\right]$, then φ is a bijection on Σ_1, Σ_2 with $\varphi(\mathcal{R}_1) = \mathcal{R}_2$ and $\varphi(l^E(\Sigma_i, \Sigma_j)) = l^E(\varphi(\Sigma_i, \Sigma_j))$ for $\forall (\Sigma_1, \Sigma_2) \in E\left(G\left[\widetilde{\Sigma}_1; \widetilde{\mathcal{R}}_1\right]\right)$.*

Similarly, we convince this definition by multi-sets. For such multi-spaces, there is a simple result on combinatorially equivalence following.

Theorem 9.1.2 *Let $\widetilde{S}_1$, $\widetilde{S}_2$ be multi-sets with $\widetilde{S}_1 = \bigcup\limits_{i=1}^{m} S_i^1$ and $\widetilde{S}_2 = \bigcup\limits_{i=1}^{m} S_i^2$. Then $\widetilde{S}_1$ is combinatorially equivalent to $\widetilde{S}_2$ if and only if there is a bijection $\sigma : \widetilde{S}_1 \to \widetilde{S}_2$ such that $\sigma(S_i^1) \in V\left(G\left[\widetilde{S}_2\right]\right)$ and $\sigma(S_1^1 \bigcap S_2^1) = \sigma(S_1^1) \bigcap \sigma(S_2^1)$, where $\sigma(S_i^1) = \{\ \sigma(e) \mid e \in S_i^1\ \}$ for any integer $i,\ 1 \leq i \leq m$.*

Proof If the multi-set $\widetilde{S}_1$ is combinatorially equivalent to that of $\widetilde{S}_2$, then there are must be a bijection $\sigma : \widetilde{S}_1 \to \widetilde{S}_2$ such that $\sigma(S_i^1) \in V\left(G\left[\widetilde{S}_2\right]\right)$ and $\sigma(S_1^1 \bigcap S_2^1) = \sigma(S_1^1) \bigcap \sigma(S_2^1)$ by Definition 9.1.2.

Conversely, if $\sigma(S_i^1) \in V\left(G\left[\widetilde{S}_2\right]\right)$ and $\sigma(S_1^1 \bigcap S_2^1) = \sigma(S_1^1) \bigcap \sigma(S_2^1)$, we are easily knowing that $\sigma : V\left(G\left[\widetilde{S}_1\right]\right) \to V\left(G\left[\widetilde{S}_2\right]\right)$ is a bijection and $(S_i^1, S_j^1) \in E\left(G\left[\widetilde{S}_1\right]\right)$ if and only if $(\sigma(S_i^1), \sigma(S_j^1)) \in E\left(G\left[\widetilde{S}_2\right]\right)$ because $\sigma(S_1^1 \bigcap S_2^1) = \sigma(S_1^1) \bigcap \sigma(S_2^1)$. Thus σ is an isomorphism from graphs $G\left[\widetilde{S}_1\right]$ to $G\left[\widetilde{S}_2\right]$ by definition. Now if $\sigma : S_i^1 \in V\left(G\left[\widetilde{S}_1\right]\right) \to S_j^2 \in V\left(G\left[\widetilde{S}_2\right]\right)$, it is clear that σ is a bijection on S_i^1, S_j^2 because σ is a bijection from $\widetilde{S}_1$ to $\widetilde{S}_2$. Applying $\sigma(S_1^1 \bigcap S_2^1) = \sigma(S_1^1) \bigcap \sigma(S_2^1)$, we are easily finding that $\sigma(l^E(S_i^1, S_j^1)) = l^E(\sigma(S_i^1, S_j^2))$ by $l^E(S_i^1, S_j^1) = S_i^1 \bigcap S_j^1$ for $\forall (S_i^1, S_j^1) \in E\left(G\left[\widetilde{S}_1\right]\right)$. So the multi-set $\widetilde{S}_1$ is combinatorially equivalent to that of $\widetilde{S}_2$. $\square$

If $\widetilde{S}_1 = \widetilde{S}_2 = \widetilde{S}$, such a combinatorial equivalence is nothing but a permutation on $\widetilde{S}$. This fact enables one to get the following conclusion.

Corollary 9.1.1 *Let* $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$ *be a multi-set with* $|\widetilde{S}| = n$, $|S_i| = n_i$, $1 \leq i \leq m$. *Then there are* $n! - \prod\limits_{i=1}^{m} n_i$ *multi-sets combinatorially equivalent to* $\widetilde{S}$ *with elements in* $\widetilde{S}$.

Proof Applying Theorem 9.1.2, all multi-sets combinatorially equivalent $\widetilde{S}$ should be $\widetilde{S}^{\varpi}$, where ϖ is a permutation on elements in $\widetilde{S}$. The number of such permutations is $n!$. It should be noted that $\widetilde{S}^{\varpi} = \widetilde{S}$ if $\varpi = \varpi_1\varpi_2 \cdots \varpi_m$, where each ϖ_i is a permutation on S_i, $1 \leq i \leq m$. Thus there are $n! - \prod\limits_{i=1}^{m} n_i$ multi-sets combinatorially equivalent to $\widetilde{S}$. $\square$

A multi-set $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$ is *exact* if $S_i = \bigcup\limits_{j=1, j\neq i}^{m} (S_j \bigcap S_i)$. For example, let $S_1 = \{a, d, e\}$, $S_2 = \{a, b, e\}$, $S_3 = \{b, c, f\}$ and $S_4 = \{c, d, f\}$. Then the multi-set $\widetilde{S} = S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4$ is exact with an inherited graph shown in Fig.9.1.2.

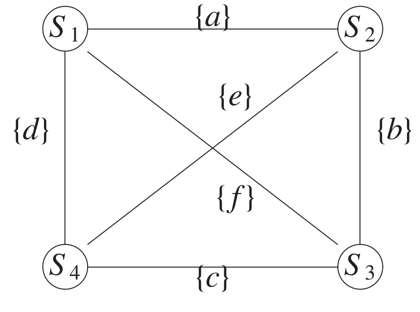


Fig.9.1.2

Then the following result is clear by the definition of exact multi-set.

Theorem 9.1.3 *An exact multi-set $\widetilde{S}$ uniquely determine an edge-labeled graph $G\left[\widetilde{S}\right]$, and conversely, an edge-labeled graph G^{l_E} also determines an exact multi-set $\widetilde{S}$ uniquely.*

Proof By Definition 9.1.1, a multi-space $\widetilde{S}$ determines an edge-labeled graph G^{l_E} uniquely. Similarly, let G^{l_E} be an edge-labeled graph. Then we are easily get an exact multi-set

$$\widetilde{S}=\bigcup_{v\in V(G^{l_E})} S_v \quad \text{with} \quad S_v=\bigcup_{e\in N_{G^{l_E}}(v)} \varpi(e). \qquad \square$$

9.1.2 Algebraic Exact Multi-System. Let $\left(\widetilde{A};\widetilde{O}\right)$ be an algebraic multi-system with $\widetilde{A}=\bigcup\limits_{i=1}^{n} A^i$ and $\widetilde{O}=\{\circ_i,\ 1\le i\le n\}$, i.e., each $(A^i;\circ_i)$ is an algebraic system for integers $1\le i\le n$. By Definition 9.1.1, we get an edge-labeled graph $G\left[\widetilde{A};\widetilde{O}\right]$ with edge labeling l_E determined by

$$l_E(A_{\circ_i},A_{\circ_j})=(A_{\circ_i}\bigcap A_{\circ_j};\{\circ_i,\circ_j\})$$

for any $(A_{\circ_i},A_{\circ_j})\in E\left(G\left[\widetilde{G};\widetilde{O}\right]\right)$, such as those shown in Fig.9.1.3, where $A_{\circ_l}=(A^l;\circ_l)$ for integers $1\le l\le n$.

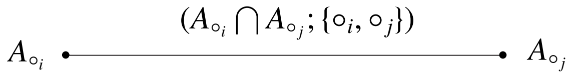


Fig.9.1.3

For determining combinatorially equivalent algebraic multi-systems, the following result is useful.

Theorem 9.1.4 *Let $\left(\widetilde{A}_1;\widetilde{O}_1\right)$, $\left(\widetilde{A}_2;\widetilde{O}_2\right)$ be algebraic multi-systems with $\widetilde{A}_1=\bigcup\limits_{i=1}^{m} A_i^1$, $\widetilde{O}_1=\{\circ_i^1,\ 1\le i\le n\}$ and $\widetilde{A}_2=\bigcup\limits_{i=1}^{m} A_i^2$, $\widetilde{O}_2=\{\circ_i^2,\ 1\le i\le n\}$. Then $\left(\widetilde{A}_1;\widetilde{O}_1\right)$ is combinatorially equivalent to $\left(\widetilde{A}_2;\widetilde{O}_2\right)$ if and only if there is a bijection $\sigma:\ \widetilde{A}_1\to\widetilde{A}_2$ such that $\sigma(A_i^1)\in V(G[\widetilde{A}_2])$ and $\sigma(A_1^1\bigcap A_2^1)=\sigma(A_1^1)\bigcap\sigma(A_2^1)$, where $\sigma(A_i^1)=\{\ \sigma(h)\mid h\in A_i^1\ \}$ for any integer i, $1\le i\le m$.*

Proof The proof is similar to that of Theorem 9.1.2. $\square$

Now let $(A;\circ)$ be an algebraic system. If there are subsystems $(A_i;\circ)\le(A;\circ)$ for integers $1\le i\le l$ such that

(1) for $\forall g \in A$, there are uniquely $a_i \in A_i$ such that $g = a_1 \circ a_2 \circ \cdots \circ a_l$;

(2) $a \circ b = b \circ a$ for $a \in A_i$ and $b \in A_j$, where $1 \leq i, j \leq s$, $i \neq j$,

then we say that $(A;\circ)$ is a direct product of $(A_i;\circ)$, denoted by $A = \bigodot\limits_{i=1}^{l} A_i$.

Let $\left(\widetilde{A};\widetilde{O}\right)$ be an algebraic multi-system with $\widetilde{A} = \bigcup\limits_{i=1}^{n} A^i$ and $\widetilde{O} = \{\circ_i,\ 1 \leq i \leq n\}$. Such an algebraic multi-system is said to be *favorable* if for any integer i, $1 \leq i \leq n$, $(A^i \bigcap A^j;\circ_i)$ is itself an algebraic system or empty set $\emptyset$ for integers $1 \leq j \leq n$. Similarly, such an algebraic multi-system is *exact* if for $\forall A_\circ \in V\left(G\left[\widetilde{A};\widetilde{O}\right]\right)$,

$$A_\circ = \bigodot_{A_\bullet \in N_{G[\widetilde{A};\widetilde{O}]}(A_\circ)} \left(A_\circ \bigcap A_\bullet\right).$$

An algebraic multi-system $\left(\widetilde{A};\widetilde{O}\right)$ with $\widetilde{A} = \bigcup\limits_{i=1}^{n} A^i$ and $\widetilde{O} = \{\circ_i,\ 1 \leq i \leq n\}$ is said to be *in-associative* if

$$(a \circ_i b) \circ_i c = a \circ_i (b \circ_i c) \quad \text{and} \quad (a \circ_j b) \circ_j c = a \circ_j (b \circ_j c)$$

hold for elements $a, b, c \in A^i \bigcap A^j$ for integers $1 \leq i, j \leq n$ providing they exist.

9.1.3 Multi-Group Underlying Graph. For favorable multi-groups, we know the following result.

Theorem 9.1.5 *A favorable multi-group is an in-associative system.*

Proof Let $\left(\widetilde{G};\widetilde{O}\right)$ be a multi-group with $\widetilde{G} = \bigcup\limits_{i=1}^{n} G^i$ and $\widetilde{O} = \{\circ_i,\ 1 \leq i \leq n\}$. Clearly, $G^i \bigcap G^j \subset G^i$ and $G^i \bigcap G^j \subset G^j$ for integers $1 \leq i, j \leq n$. Whence, the associative laws hold for elements in $(G^i \bigcap G^j;\circ_i)$ and $(G^i \bigcap G^j;\circ_j)$. Thus $(G^i \bigcap G^j;\{\circ_i,\circ_j\})$ is an in-associative system for integers $1 \leq i, j \leq n$ by definition. □

Particularly, if $\circ_i = \circ$, i.e., $(G^i;\circ)$ is a subgroup of a group for integers $1 \leq i \leq n$ in Theorem 9.1.5, we get the following conclusion.

Corollary 9.1.2 *Let $(G^i;\circ)$ be subgroups of a group $(\mathscr{G};\circ)$ for integers $1 \leq i \leq n$. Then a multi-group $(\widetilde{G};\{\circ\})$ with $\widetilde{G} = \bigcup\limits_{i=1}^{n} G^i$ is favorable if and only if $(G^i \bigcap G^j;\{\circ\})$ is a subgroup of group $(\mathscr{G};\circ)$ for any integers $1 \leq i, j \leq n$, i.e., $G\left[\widetilde{G};\{\circ\}\right] \simeq K_n$.*

Proof Applying Corollary 1.2.1 with $G^i \bigcap G^j \supseteq \{1_{\widetilde{G}}\}$ for any integers $1 \leq i, j \leq n$, we know this conclusion. □

Applying Theorem 9.1.4, we have the following conclusion on combinatorially equivalent multi-groups.

Theorem 9.1.6 *Let* $\left(\widetilde{G}_1;\widetilde{O}_1\right)$*,* $\left(\widetilde{G}_2;\widetilde{O}_2\right)$ *be multi-groups with* $\widetilde{G}_1 = \bigcup\limits_{i=1}^{n} G_i^1$*,* $\widetilde{O}_1 = \{\circ_i^1,\ 1 \leq i \leq n\}$ *and* $\widetilde{G}_2 = \bigcup\limits_{i=1}^{n} G_i^2$*,* $\widetilde{O}_2 = \{\circ_i^2,\ 1 \leq i \leq n\}$*. Then* $\left(\widetilde{G}_1;\widetilde{O}_1\right)$ *is combinatorially equivalent to* $\left(\widetilde{G}_2;\widetilde{O}_2\right)$ *if and only if there is a bijection* $\phi : \widetilde{G}_1 \to \widetilde{G}_2$ *such that* $\phi(G_i^1) \in V\left(G\left[\widetilde{G}_2;\widetilde{O}_2\right]\right)$ *is an isomorphism and* $\phi(G_1^1 \bigcap G_2^1) = \phi(G_1^1) \bigcap \phi(G_2^1)$ *for any integer* $i,\ 1 \leq i \leq n$*.*

By Theorem 9.1.3, we have known that an edge-labeled graph G^{l_E} uniquely determines an exact multi-set $\widetilde{S}$. The following result shows when such a multi-system is a multi-group.

Theorem 9.1.7 *Let* $(\widetilde{G};\widetilde{O})$ *be a favorable exact multi-system determined by an edge-labeled graph* G^{l_E} *with* $\widetilde{G} = \bigcup\limits_{u \in V(G^{l_E})} G_u$*, where* $G_u = \bigodot\limits_{v \in N_{G^{l_E}}(u)} l^E(u,v)$ *and* $\widetilde{O} = \{\circ_i,\ 1 \leq i \leq n\}$*. Then it is a multi-group if and only if for* $\forall u \in V(G^{l_E})$*, there is an operation* $\circ_u$ *in* $l_E(u,v)$ *for all* $v \in N_{G^{l_E}}(u)$ *such that for* $\forall a \in l_E(u,v_1)$*,* $b \in l_E(u,v_2)$*, there is a* $a \circ_u b^{-1} \in l_E(u,v_3)$*, where* $v_1,v_2,v_3 \in N_{G^{l_E}}(u)$*.*

Proof Clearly, if $(\widetilde{G};\widetilde{O})$ is a multi-group, then for $\forall u \in V(G^{l_E})$, there is an operation $\circ_u \in G_u$ for all $v \in N_{G^{l_E}}(u)$ such that for $\forall a \in l_E(u,v_1)$, $b \in l_E(u,v_2)$, there is $a \circ_u b^{-1} \in l_E(u,v_3)$, where $v_1,v_2,v_3 \in N_{G^{l_E}}(u)$.

Conversely, let $u \in V(G^{l_E})$. We prove that the pair $(G_u;\circ_u)$ with $G_u = \bigodot\limits_{v \in N_{G^{l_E}}(u)} l^E(u,v)$ is a group. In fact,

(1) There exists an $h \in G_u$ and $1_{G_u} = h \circ h^{-1} \in G_u$;

(2) If $a,b \in G_u$, then $a^{-1} = 1_{G_u} \circ_u a^{-1} \in G_u$. Thus $a \circ_u (b^{-1})^{-1} = a \circ_u b \in G_u$;

(3) Notice that

$$g \circ_u h = \prod_{v \in N_{G^{l_E}}(u)} g_v \circ_u h_v,$$

where $g = \prod\limits_{v \in N_{G^{l_E}}(u)} g_v \in G_u$, $h = \prod\limits_{v \in N_{G^{l_E}}(u)} h_v \in G_u$ because of $G_u = \bigodot\limits_{v \in N_{G^{l_E}}(u)} l^E(u,v)$. We know that the associative law $a \circ_u (b \circ_u c) = (a \circ_u b) \circ_u c$ for $a,b,c \in G_u$ holds by Theorem 9.1.4. Thus $(G_u;\circ_u)$ is a group for $\forall u \in V(G^{l_E})$.

Consequently, $(\widetilde{G};\widetilde{O})$ is a multi-group, □

Let $\widetilde{O} = \{\circ\}$ in Theorem 9.1.7. We get the following conclusions.

Corollary 9.1.3 *Let $\left(\widetilde{G};\{\circ\}\right)$ be an exact multi-system determined by an edge-labeled graph G^{l_E} with subgroups $l^E(u,v)$ of group $(\mathscr{G};\circ)$ for $(u,v)\in E(G^{l_E})$ such that $l^E(u,v_1)\bigcap l^E(u,v_2)=\{1_{\mathscr{G}}\}$ for $\forall(u,v_1),(u,v_2)\in E(G^{l_E})$. Then $\left(\widetilde{G};\{\circ\}\right)$ is a multi-group.*

Particularly, let $(\mathscr{G};\circ)$ be Abelian. Then we get an interesting result following by applying the fundamental theorem of finite Abelian group.

Corollary 9.1.4 *Let $\left(\widetilde{G};\{\circ\}\right)$ be an exact multi-system determined by an edge-labeled graph G^{l_E} with cyclic p-groups $l^E(u,v)$ of a finite Abelian group $(\mathscr{G};\circ)$ for $(u,v)\in E(G^{l_E})$ such that $l^E(u,v_1)\bigcap l^E(u,v_2)=\{1_{\mathscr{G}}\}$ for $\forall(u,v_1),(u,v_2)\in E(G^{l_E})$. Then $\left(\widetilde{G};\{\circ\}\right)$ is a finite Abelian multi-group, i.e., each $(G_u,\circ)$ is a finite Abelian group for $u\in V(G^{l_E})$.*

9.1.4 Multi-Ring Underlying Graph. A multi-system $\left(\widetilde{A};O_1\bigcup O_2\right)$ with $\widetilde{A}=\bigcup\limits_{i=1}^{n}A^i$, $O_1=\{\cdot_i;1\le i\le n\}$ and $O_2=\{+_i;1\le i\le n\}$ is *in-distributed* if for any integer i, $1\le i\le n$, $a\cdot_i(b+_ic)=a\cdot_ib+_ia\cdot_ic$ hold for $\forall a,b,c\in A^i\bigcap A^j$ providing they exist, usually denoted by $\left(\widetilde{A};O_1\hookrightarrow O_2\right)$. For favorable multi-rings, we know the following result.

Theorem 9.1.8 *A favorable multi-ring is an in-associative and in-distributed multi-system.*

Proof Let $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$ be a favorable multi-ring with $\widetilde{R}=\bigcup\limits_{i=1}^{n}R_i$, $O_1=\{\cdot_i;1\le i\le n\}$ and $O_2=\{+_i;1\le i\le n\}$. Notice that $R_i\bigcap R_j\subset R_i$, $R_i\bigcap R_j\subset R_j$ and $(R_i;\cdot_i,+_i)$, $(R_j;\cdot_j,+_j)$ are rings for integers $1\le i,j\le n$. Whence, if $R_i\bigcap R_j\neq\emptyset$ for integers $1\le i,j\le n$, let $a,b,c\in R_i\bigcap R_j$. Then we get that

$$(a\cdot_i b)\cdot_i c=a\cdot_i(b\cdot_i c),\qquad (a\cdot_j b)\cdot_j c=a\cdot_j(b\cdot_j c)$$
$$(a+_i b)+_i c=a+_i(b+_i c),\quad (a+_j b)+_j c=a+_j(b+_j c)$$

and

$$a\cdot_i(b+_ic)=a\cdot_ib+_ia\cdot_ic,\quad a\cdot_j(b+_jc)=a\cdot_jb+_ja\cdot_jc.$$

Thus $\left(\widetilde{R};O_1\hookrightarrow O_2\right)$ is in-associative and in-distributed. □

Particularly, if $\cdot_i=\cdot$ and $+_i=+$ for integers $1\le i\le n$ in Theorem 9.1.8, we get a conclusion following for characterizing favorable multi-rings.

Corollary 9.1.5 *Let $(R^i;\cdot,+)$ be subrings of ring a $(R;\cdot,+)$ for integers $1\le i\le n$. Then a multi-ring $(\widetilde{R};\{\cdot\}\hookrightarrow\{+\})$ with $\widetilde{R}=\bigcup\limits_{i=1}^{n}R^i$ is favorable if and only if $(R^i\bigcap R^j;\cdot,+)$ is a subring of ring $(R;\cdot,+)$ for any integers $1\le i,j\le n$, i.e., $G\left[\widetilde{R};\{\cdot\}\hookrightarrow\{+\}\right]\simeq K_n$.*

Proof Applying Theorems 1.3.2 and 9.1.8 with $R^i \bigcap R^j \supseteq \{0_+\}$ for any integers $1 \leq i, j \leq n$, we are easily knowing that $\left(\widetilde{R}; \{\cdot\} \hookrightarrow \{+\}\right)$ is favorable if and only if $(R^i \bigcap R^j; \cdot, +)$ is a subring of ring $(R; \cdot, +)$ for any integers $1 \leq i, j \leq n$ and $G\left[\widetilde{R}; \{\cdot\} \hookrightarrow \{+\}\right] \simeq K_n$. □

Similarly, we know the following result for combinatorially equivalent multi-rings by Theorem 9.1.6.

Theorem 9.1.9 *Let* $\left(\widetilde{R}^1; O_1^1 \hookrightarrow O_2^1\right)$, $\left(\widetilde{R}^2; O_1^2 \hookrightarrow O_2^2\right)$ *be multi-rings with* $\widetilde{R}^1 = \bigcup\limits_{i=1}^{n} R_i^1$, $\widetilde{O}_1 = \{\cdot_i^1,\ 1 \leq i \leq n\}$ *and* $\widetilde{R}^2 = \bigcup\limits_{i=1}^{n} G_i^2$, $\widetilde{O}_2 = \{\cdot_i^2,\ 1 \leq i \leq n\}$. *Then* $\left(\widetilde{R}^1; O_1^1 \hookrightarrow O_2^1\right)$ *is combinatorially equivalent to* $\left(\widetilde{R}^2; O_1^2 \hookrightarrow O_2^2\right)$ *if and only if there is a bijection* $\varphi: \widetilde{R}^1 \to \widetilde{R}^2$ *such that* $\varphi(R_i^1) \in V\left(G\left[\widetilde{R}^2; O_1^2 \hookrightarrow O_2^2\right]\right)$ *is an isomorphism and* $\varphi(R_1^1 \bigcap R_2^1) = \varphi(R_1^1) \bigcap \phi(R_2^1)$ *for any integer* i, $1 \leq i \leq n$.

Let $(R_1, \cdot, +), (R_2, \cdot, +), \cdot, (R_l, \cdot, +)$ be l rings. Then we get a direct sum

$$R = R_1 \bigoplus R_2 \bigoplus \cdots \bigoplus R_l$$

by the definition of direct product of additive groups $(R_i; +)$, $1 \leq i \leq l$. Define

$$(a_1, a_2, \cdots, a_l) \cdot (b_1, b_2, \cdots, b_l) = (a_1 \cdot b_1, a_2 \cdot b_2, \cdots, a_l \cdot b_l)$$

for $(a_1, a_2, \cdots, a_l), (b_1, b_2, \cdots, b_l) \in R$. Then it is easily to verify that $(R; \cdot, +)$ is also a ring. Such a ring is called the direct sum of rings $(R_1, \cdot, +), (R_2, \cdot, +), \cdot, (R_l, \cdot, +)$, denoted by $R = \bigoplus\limits_{i=1}^{l} R^i$.

A multi-ring $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ with $\widetilde{R} = \bigcup\limits_{i=1}^{n} R_i$, $O_1 = \{\cdot_i; 1 \leq i \leq n\}$ and $O_2 = \{+_i; 1 \leq i \leq n\}$ is said to be *exact* if it is favorable and $R_i = \bigoplus\limits_{j=1}^{n} (R_i \bigcap R_j)$ for any integer i $1 \leq i \leq n$. Thus $R_i = \bigoplus\limits_{(R_i, R_j) \in E(G[\widetilde{R}; O_1 \hookrightarrow O_2])} \varpi(R_i, R_j)$ in its inherited graph $G^{l_E}\left[\widetilde{R}; O_1 \hookrightarrow O_2\right]$. The following result is an immediately consequence of Theorem 9.1.7.

Theorem 9.1.10 *Let* $\left(\widetilde{R}; O_1 \hookrightarrow O_2\right)$ *be a favorable exact multi-system determined by an edge-labeled graph* G^{l_E} *with* $\widetilde{R} = \bigcup\limits_{u \in V(G^{l_E})} R_u$, *where* $R_u = \bigoplus\limits_{v \in N_{G^{l_E}}(u)} l^E(u, v)$, $O_1 = \{\cdot_i,\ 1 \leq i \leq n\}$ *and* $O_2 = \{+_i,\ 1 \leq i \leq n\}$. *Then it is a multi-ring if and only if for* $\forall u \in V(G^{l_E})$, *there are two operations* $+_u$, $\cdot_u$ *in* $l_E(u, v)$ *for all* $v \in N_{G^{l_E}}(u)$ *such that for* $\forall a \in l_E(u, v_1)$, $b \in l_E(u, v_2)$, *there is* $a -_u b \in l_E(u, v_3)$ *and* $a \cdot_u b \in l_E(u, v_4)$, *where* $v_1, v_2, v_3, v_4 \in N_{G^{l_E}}(u)$.

Particularly, if $\cdot_i = \cdot$ and $+_i = +$ for integers $1 \leq i \leq n$ in Theorem 9.1.10, we get the following consequence.

Corollary 9.1.6 *Let* $\left(\widetilde{R}; \{\cdot\} \hookrightarrow \{+\}\right)$ *be an exact multi-system determined by an edge-labeled graph* G^{l_E} *with subrings* $l^E(u,v)$ *of a ring* $(R;\cdot,+)$ *for* $(u,v) \in E(G^{l_E})$ *such that* $l^E(u,v_1) \bigcap l^E(u,v_2) = \{0_+\}$ *for* $\forall (u,v_1),(u,v_2) \in E(G^{l_E})$. *Then* $\left(\widetilde{R}; \{\cdot\} \hookrightarrow \{+\}\right)$ *is a multi-ring.*

Let p_i, $1 \leq i \leq s$ be different prime numbers. Then each $(p_i\mathbb{Z};\cdot,+)$ is a subring of the integer ring $(\mathbb{Z};\cdot,+)$ such that $(p_i\mathbb{Z})\bigcap(p_j\mathbb{Z}) = \{0\}$. Thus such subrings satisfy the conditions of Corollary 9.1.6, which enables one to get an edge-labeled graph with its correspondent exact multi-ring. For example, such an edge-labeled graph is shown in Fig.9.1.4 for $n = 6$.

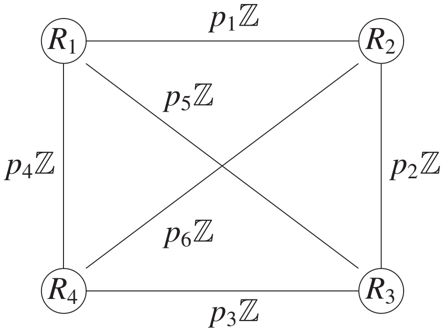


Fig.9.1.4

9.1.5 Vector Multi-Space Underlying Graph. According to Theorem 1.4.6, two vector spaces V_1 and V_2 over a field F are isomorphic if and only if $\dim V_1 = \dim V_2$. This fact enables one to characterize a vector space by its basis. Let $\left(\widetilde{V}; \widetilde{F}\right)$ be vector multi-space. Choose the edge labeling $l_E : (V_u, V_v) \to \mathscr{B}(V_u \bigcap V_v)$ for $\forall (V_u, V_v) \in E\left(G\left[\widetilde{V}\right]\right)$ in Definition 9.1.1, where $\mathscr{B}(V_u \bigcap V_v)$ denotes the basis of vector space $V_u \bigcap V_v$, such as those shown in Fig.9.1.5.

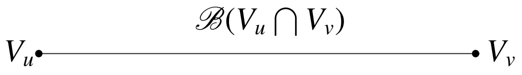


Fig.9.1.5

Let $A \subset B$. An inclusion mapping $\iota : A \to B$ is such a $1-1$ mapping that $\iota(a) = a$ for $\forall a \in B$ if $a \in A$. The next result combinatorially characterizes vector multi-subspaces.

Theorem 9.1.11 *Let* $\left(\widetilde{V}_1;\widetilde{F}_1\right)$ *and* $\left(\widetilde{V}_2;\widetilde{F}_2\right)$ *be vector multi-spaces with* $\widetilde{V}_1 = \bigcup\limits_{i=1}^{n} V_i^1$ *and* $\widetilde{F}_1 = \bigcup\limits_{i=1}^{n} F_i^1$, *and* $\widetilde{V}_2 = \bigcup\limits_{i=1}^{n} V_i^2$ *and* $\widetilde{F}_1 = \bigcup\limits_{i=1}^{n} F_i^2$. *Then* $\left(\widetilde{V}_1;\widetilde{F}_1\right)$ *is a vector multi-subspace of* $\left(\widetilde{V}_2;\widetilde{F}_2\right)$ *if and only if there is an inclusion* $\iota : \widetilde{V}_1 \to \widetilde{V}_2$ *such that* $\iota\left(G\left[\widetilde{V}_1;\widetilde{F}_1\right]\right) \prec G\left[\widetilde{V}_2;\widetilde{F}_2\right]$.

Proof If $\left(\widetilde{V}_1;\widetilde{F}_1\right)$ is a vector multi-subspace of $\left(\widetilde{V}_2;\widetilde{F}_2\right)$, by definition there are must be $V_i^1 \subset V_{i_j}^2$ and $F_i^1 \subset F_{i_j}^2$ for integers $1 \leq i \leq n$, where $i_j \in \{1, 2, \cdots, n\}$. Then there is an inclusion mapping $\iota : \widetilde{V}_1 \to \widetilde{V}_2$ determined by $\iota(V_i^1) = V_i^1 \subset V_{i_j}^2$ such that $\iota\left(G\left[\widetilde{V}_1;\widetilde{F}_1\right]\right) \prec G\left[\widetilde{V}_2;\widetilde{F}_2\right]$.

Conversely, if there is an inclusion $\iota : \widetilde{V}_1 \to \widetilde{V}_2$ such that $\iota\left(G\left[\widetilde{V}_1;\widetilde{F}_1\right]\right) \prec G\left[\widetilde{V}_2;\widetilde{F}_2\right]$, then there must be $\iota(V_i^1) \subset V_{j_i}^2$ and $\iota(F_i^1) \subset F_{j_i}^2$ for some integers $j_i \in \{1, 2, \cdots, n\}$. Thus $\widetilde{V}_1 = \bigcup\limits_{i=1}^{n} V_i^1 = \bigcup\limits_{i=1}^{n} \iota(V_i^1) \subset \bigcup\limits_{j=1}^{n} V_{i_j}^2 = \widetilde{V}_2$ and $\widetilde{F}_1 = \bigcup\limits_{i=1}^{n} F_i^1 = \bigcup\limits_{i=1}^{n} \iota(F_i^1) \subset \bigcup\limits_{j=1}^{n} F_{i_j}^2 = \widetilde{F}_2$, i.e., $\left(\widetilde{V}_1;\widetilde{F}_1\right)$ is a vector multi-subspace of $\left(\widetilde{V}_2;\widetilde{F}_2\right)$. □

Let V be a vector space and let $V_1, V_2 \subset V$ be two vector subspaces. For $\forall \overline{a} \in V$, if there are vectors $\overline{b} \in V_1$ and $\overline{c} \in V_2$ such that $\overline{a} = \overline{b} + \overline{c}$ is uniquely, then V is said a *direct sum* of V_1 and V_2, denoted by $V = V_1 \bigoplus V_2$. It is easily to show that if $V_1 \bigcap V_2 = \overline{0}$, then $V = V_1 \bigoplus V_2$.

A vector multi-space $(\widetilde{V};\widetilde{F})$ with $\widetilde{V} = \bigcup\limits_{i=1}^{n} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{n} F_i$ is said to be *exact* if

$$V_i = \bigoplus_{j \neq i} (V_i \bigcap V_j)$$

holds for integers $1 \leq i \leq n$. We get a necessary and sufficient condition for exact vector multi-spaces following.

Theorem 9.1.12 *Let* $\left(\widetilde{V};\widetilde{F}\right)$ *be a vector multi-space with* $\widetilde{V} = \bigcup\limits_{i=1}^{n} V_i$ *and* $\widetilde{F} = \bigcup\limits_{i=1}^{n} F_i$. *Then it is exact if and only if*

$$\mathscr{B}(V) = \bigcup_{(V,V') \in E(G[\widetilde{V};\widetilde{F}])} \mathscr{B}\left(V \bigcap V'\right) \quad \textit{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};\widetilde{F}]}(V)$.

Proof If $\left(\widetilde{V};\widetilde{F}\right)$ is exact, i.e., $V_i = \bigoplus\limits_{j \neq i} (V_i \bigcap V_j)$, then it is clear that

$$\mathscr{B}(V) = \bigcup_{(V,V') \in E\left(G\left[\widetilde{V};\widetilde{F}\right]\right)} \mathscr{B}\left(V \bigcap V'\right) \quad \text{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

by the fact that $(V, V') \in E\left(G\left[\widetilde{V}; \widetilde{F}\right]\right)$ if and only if $V \bigcap V' \neq \emptyset$ for $\forall V', V'' \in N_{G[\widetilde{V};\widetilde{F}]}(V)$ by definition.

Conversely, if

$$\mathscr{B}(V) = \bigcup_{(V,V') \in E(G[\widetilde{V};\widetilde{F}])} \mathscr{B}\left(V \bigcap V'\right) \quad \text{and} \quad \mathscr{B}\left(V \bigcap V'\right) \bigcap \mathscr{B}\left(V \bigcap V''\right) = \emptyset$$

for $V', V'' \in N_{G[\widetilde{V};\widetilde{F}]}(V)$, notice also that $(V, V') \in E\left(G\left[\widetilde{V}; \widetilde{F}\right]\right)$ if and only if $V \bigcap V' \neq \emptyset$, we know that

$$V_i = \bigoplus_{j \neq i} \left(V_i \bigcap V_j\right)$$

for integers $1 \leq i \leq n$ by Theorem 1.4.4. This completes the proof. □

§9.2 COMBINATORIAL EUCLIDEAN SPACES

9.2.1 Euclidean Space. A *Euclidean space* on a real vector space $\mathbf{E}$ over a field F is a mapping

$$\langle \cdot, \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbf{R} \ \text{ with } \ (\overline{e}_1, \overline{e}_2) \to \langle \overline{e}_1, \overline{e}_2 \rangle \ \text{ for} \forall \overline{e}_1, \overline{e}_2 \in \mathbf{E}$$

such that for $\overline{e}, \overline{e}_1, \overline{e}_2 \in \mathbf{E}$, $\alpha \in F$

(E1) $\langle \overline{e}, \overline{e}_1 + \overline{e}_2 \rangle = \langle \overline{e}, \overline{e}_1 \rangle + \langle \overline{e}, \overline{e}_2 \rangle$;

(E2) $\langle \overline{e}, \alpha \overline{e}_1 \rangle = \alpha \langle \overline{e}, \overline{e}_1 \rangle$;

(E3) $\langle \overline{e}_1, \overline{e}_2 \rangle = \langle \overline{e}_2, \overline{e}_1 \rangle$;

(E4) $\langle \overline{e}, \overline{e} \rangle \geq 0$ and $\langle \overline{e}, \overline{e} \rangle = 0$ if and only if $\overline{e} = \overline{0}$.

In a Euclidean space $\mathbf{E}$, the number $\sqrt{\langle \overline{e}, \overline{e} \rangle}$ is called its *norm*, and denoted by $\|\overline{e}\|$. It can be shown that

(1) $\left\langle \overline{0}, \overline{e} \right\rangle = \left\langle \overline{e}, \overline{0} \right\rangle = 0$ for $\forall \overline{e} \in \mathbf{E}$;

(2) $\left\langle \sum\limits_{i=1}^{n} x_i \overline{e}_i^1, \sum\limits_{j=1}^{m} y_i \overline{e}_j^2 \right\rangle = \sum\limits_{i=1}^{n} \sum\limits_{i=1}^{m} x_i y_j \left\langle \overline{e}_i^1, \overline{e}_j^2 \right\rangle$, for $\overline{e}_i^s \in \mathbf{E}$, where $1 \leq i \leq \max\{m, n\}$ and $s = 1$ or 2.

In fact, let $\overline{e}_1 = \overline{e}_2 = \overline{0}$ in (E1). Then $\left\langle \overline{e}, \overline{0} \right\rangle = 0$. Applying (E3), we get that $\left\langle \overline{0}, \overline{e} \right\rangle = 0$. This is the formula in (1). For the equality (2), applying conditions (E1)-(E2), we know that

$$\left\langle \sum_{i=1}^{n} x_i \overline{e}_i^1, \sum_{j=1}^{m} y_i \overline{e}_j^2 \right\rangle = \sum_{j=1}^{m} \left\langle \sum_{i=1}^{n} x_i \overline{e}_i^1, y_i \overline{e}_j^2 \right\rangle = \sum_{j=1}^{m} y_i \left\langle \sum_{i=1}^{n} x_i \overline{e}_i^1, \overline{e}_j^2 \right\rangle$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} y_i \left\langle \overline{e}_j^2, \sum_{i=1}^{n} x_i \overline{e}_i^1 \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_i \left\langle \overline{e}_j^2, \overline{e}_i^1 \right\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_i \left\langle \overline{e}_i^1, \overline{e}_j^2 \right\rangle .
\end{aligned}
$$

Theorem 9.2.1 *Let* **E** *be a Euclidean space. Then for* $\forall \overline{e}_1, \overline{e}_2 \in \mathbf{E}$,

(1) $|\langle \overline{e}_1, \overline{e}_2 \rangle| \leq ||\overline{e}_1|| ||\overline{e}_2||$;

(2) $||\overline{e}_1 + \overline{e}_2|| \leq ||\overline{e}_1|| + ||\overline{e}_2||$.

Proof Notice that the inequality (1) is hold if $\overline{e}_1$ or $\overline{e}_2 = \overline{0}$. Assume $\overline{e}_1 \neq \overline{0}$. Let $x = \dfrac{\langle \overline{e}_1, \overline{e}_2 \rangle}{\langle \overline{e}_1, \overline{e}_1 \rangle}$. Since

$$\langle \overline{e}_2 - x\overline{e}_1, \overline{e}_2 - x\overline{e}_1 \rangle = \langle \overline{e}_2, \overline{e}_2 \rangle - 2\langle \overline{e}_1, \overline{e}_2 \rangle x + \langle \overline{e}_1, \overline{e}_1 \rangle x^2 \geq 0.$$

Replacing x by $\dfrac{\langle \overline{e}_1, \overline{e}_2 \rangle}{\langle \overline{e}_1, \overline{e}_1 \rangle}$ in it, then

$$\langle \overline{e}_1, \overline{e}_1 \rangle \langle \overline{e}_2, \overline{e}_2 \rangle - \langle \overline{e}_1, \overline{e}_2 \rangle^2 \geq 0.$$

Whence, we get that

$$|\langle \overline{e}_1, \overline{e}_2 \rangle| \leq ||\overline{e}_1|| ||\overline{e}_2||.$$

For the inequality (2), applying inequality (1), we know that

$$
\begin{aligned}
||\langle \overline{e}_1, \overline{e}_2 \rangle||^2 &= \langle \overline{e}_1 + \overline{e}_2, \overline{e}_1 + \overline{e}_2 \rangle \\
&= \langle \overline{e}_1, \overline{e}_1 \rangle + 2\langle \overline{e}_1, \overline{e}_2 \rangle + \langle \overline{e}_2, \overline{e}_2 \rangle \\
&= \langle \overline{e}_1, \overline{e}_1 \rangle + 2|\langle \overline{e}_1, \overline{e}_2 \rangle| + \langle \overline{e}_2, \overline{e}_2 \rangle \\
&\leq \langle \overline{e}_1, \overline{e}_1 \rangle + 2||\langle \overline{e}_1, \overline{e}_1 \rangle|| ||\langle \overline{e}_2, \overline{e}_1 \rangle|| + \langle \overline{e}_2, \overline{e}_2 \rangle \\
&= (||\overline{e}_1|| + ||\overline{e}_2||)^2.
\end{aligned}
$$

Thus

$$||\overline{e}_1 + \overline{e}_2|| \leq ||\overline{e}_1|| + ||\overline{e}_2||. \qquad \square$$

Let **E** be a Euclidean space, $\overline{a}, \overline{b} \in \mathbf{E}$, $\overline{a} \neq \overline{0}$, $\overline{b} \neq \overline{0}$. The *angle* between $\overline{a}$ and $\overline{b}$ is defined by

$$\cos\theta = \frac{\left\langle \overline{a}, \overline{b} \right\rangle}{||\overline{a}|| ||\overline{b}||}.$$

Notice that by Theorem 9.2.1(1), the inequality

$$-1 \leq \frac{\left\langle \overline{a}, \overline{b} \right\rangle}{\|\overline{a}\|\|\overline{b}\|} \leq -1$$

always holds. Thus the angle between $\overline{a}$ and $\overline{b}$ is well-defined. Let $\overline{x}, \overline{y} \in \mathbf{E}$. Call $\overline{x}$ and $\overline{y}$ to be *orthogonal* if $\langle \overline{x}, \overline{y} \rangle = 0$. For a basis $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m$ of **E** if $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m$ are orthogonal two by two, such a basis is called an *orthogonal basis*. Furthermore, if $\|\overline{e}_i\| = 1$ for integers $1 \leq i \leq m$, an orthogonal basis $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m$ is called a *normal basis*.

Theorem 9.2.2 *Any n-dimensional Euclidean space* **E** *has an orthogonal basis.*

Proof Let $\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_n$ be a basis of **E**. We construct an orthogonal basis $\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_n$ of this space. Notice that $\left\langle \overline{b}_1, \overline{b}_1 \right\rangle \neq 0$. Choose $\overline{b}_1 = \overline{a}_1$ and let

$$\overline{b}_2 = \overline{a}_2 - \frac{\left\langle \overline{a}_2, \overline{b}_1 \right\rangle}{\left\langle \overline{b}_1, \overline{b}_1 \right\rangle}\overline{b}_1.$$

Then $\overline{b}_2$ is a linear combination of $\overline{a}_1$ and $\overline{a}_2$ and

$$\left\langle \overline{b}_2, \overline{b}_1 \right\rangle = \left\langle \overline{a}_2, \overline{b}_1 \right\rangle - \frac{\left\langle \overline{a}_2, \overline{b}_1 \right\rangle}{\left\langle \overline{b}_1, \overline{b}_1 \right\rangle}\left\langle \overline{b}_1, \overline{b}_1 \right\rangle = 0,$$

i.e., $\overline{b}_2$ is orthogonal with $\overline{b}_1$.

If we have constructed $\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_k$ for an integer $1 \leq k \leq n-1$, and each of them is a linear combination of $\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_i$, $1 \leq i \leq k$. Notice $\left\langle \overline{b}_1, \overline{b}_1 \right\rangle, \left\langle \overline{b}_2, \overline{b}_2 \right\rangle, \cdots, \left\langle \overline{b}_{k-1}, \overline{b}_{k-1} \right\rangle \neq 0$. Let

$$\overline{b}_k = \overline{a}_k - \frac{\left\langle \overline{a}_k, \overline{b}_1 \right\rangle}{\left\langle \overline{b}_1, \overline{b}_1 \right\rangle}\overline{b}_1 - \frac{\left\langle \overline{a}_k, \overline{b}_2 \right\rangle}{\left\langle \overline{b}_2, \overline{b}_2 \right\rangle}\overline{b}_2 - \cdots - \frac{\left\langle \overline{a}_k, \overline{b}_{k-1} \right\rangle}{\left\langle \overline{b}_{k-1}, \overline{b}_{k-1} \right\rangle}\overline{b}_{k-1}.$$

Then $\overline{b}_k$ is a linear combination of $\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_{k-1}$ and

$$\begin{aligned}\left\langle \overline{b}_k, \overline{b}_i \right\rangle &= \left\langle \overline{a}_k, \overline{b}_i \right\rangle - \frac{\left\langle \overline{a}_k, \overline{b}_1 \right\rangle}{\left\langle \overline{b}_1, \overline{b}_1 \right\rangle}\left\langle \overline{b}_1, \overline{b}_i \right\rangle - \cdots - \frac{\left\langle \overline{a}_k, \overline{b}_{k-1} \right\rangle}{\left\langle \overline{b}_{k-1}, \overline{b}_{k-1} \right\rangle}\left\langle \overline{b}_{k-1}, \overline{b}_i \right\rangle \\ &= \left\langle \overline{a}_k, \overline{b}_i \right\rangle - \frac{\left\langle \overline{a}_k, \overline{b}_i \right\rangle}{\left\langle \overline{b}_i, \overline{b}_i \right\rangle}\left\langle \overline{b}_i, \overline{b}_i \right\rangle = 0\end{aligned}$$

for $i = 1, 2, \cdots, k-1$. Apply the induction principle, this proof is completes. □

Corollary 9.2.1 *Any n-dimensional Euclidean space* **E** *has a normal basis.*

Proof According to Theorem 9.2.2, any n-dimensional Euclidean space $\mathbf{E}$ has an orthogonal basis $\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_m$. Now let $\overline{e}_1 = \dfrac{\overline{a}_1}{\|\overline{a}_1\|}$, $\overline{e}_2 = \dfrac{\overline{a}_2}{\|\overline{a}_2\|}$, $\cdots$, $\overline{e}_m = \dfrac{\overline{a}_m}{\|\overline{a}_m\|}$. Then we find that

$$\left\langle \overline{e}_i, \overline{e}_j \right\rangle = \frac{\left\langle \overline{a}_i, \overline{a}_j \right\rangle}{\|\overline{a}_i\|\|\overline{a}_j\|} = 0 \ \text{ and } \ \|\overline{e}_i\| = \|\frac{\overline{a}_i}{\|\overline{a}_i\|}\| = \frac{\|\overline{a}_i\|}{\|\overline{a}_i\|} = 1$$

for integers $1 \leq i, j \leq m$ by definition. Thus $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m$ is a normal basis. □

9.2.2 Combinatorial Euclidean Space. Let $\mathbf{R}^n$ be a Euclidean space with normal basis $\mathscr{B}(\mathbf{R}^n) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_n\}$, where $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$, namely, it has n orthogonal orientations. Generally, we think any Euclidean space $\mathbf{R}^n$ is a subspace of Euclidean space $\mathbf{R}^{n_\infty}$ with a finite but sufficiently large dimension n_∞, then two Euclidean spaces $\mathbf{R}^{n_u}$ and $\mathbf{R}^{n_v}$ have a non-empty intersection if and only if they have common orientations.

A *combinatorial Euclidean space* is a geometrical multi-space $(\widetilde{\mathbf{R}}; \mathcal{R})$ with $\widetilde{\mathbf{R}} = \bigcup\limits_{i=1}^{m} \mathbf{R}^{n_i}$ underlying an edge-labeled graph G^{l_E} with edge labeling

$$l_E : \ (\mathbf{R}^{n_i}, \mathbf{R}^{n_j}) \to \mathscr{B}\left(\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j}\right)$$

for $\forall(\mathbf{R}^{n_i}, \mathbf{R}^{n_j}) \in E(G^{l_E})$, where $\mathcal{R}$ consists of Euclidean axioms, usually abbreviated to $\widetilde{\mathbf{R}}$. For example, a combinatorial Euclidean space $(\widetilde{\mathbf{R}}; \mathcal{R})$ is shown by edge-labeled graph G^{l_E} in Fig.9.2.1,

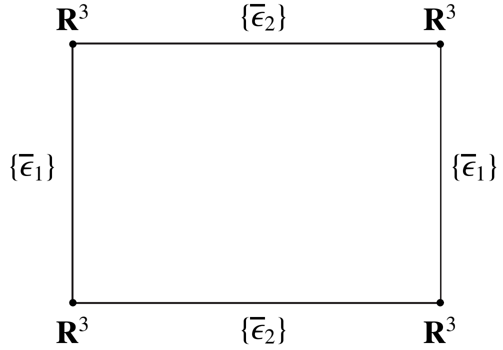

$\mathbf{R}^3$
$\{\overline{\epsilon}_2\}$
$\mathbf{R}^3$
$\{\overline{\epsilon}_1\}$
$\{\overline{\epsilon}_1\}$
$\mathbf{R}^3$
$\{\overline{\epsilon}_2\}$
$\mathbf{R}^3$

Fig.9.2.1

We are easily to know that $\mathscr{B}(\widetilde{\mathbf{R}}) = \{\overline{\epsilon}_1, \overline{\epsilon}_2, \overline{\epsilon}_3, \overline{\epsilon}_4, \overline{\epsilon}_5, \overline{\epsilon}_6\}$, i.e., $\dim\widetilde{\mathbf{R}} = 6$. Generally, we can determine the dimension of a combinatorial Euclidean space by its underlying structure G^{l_E} following.

Theorem 9.2.3 *Let $\widetilde{\mathbf{R}}$ be a combinatorial Euclidean space consisting of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure G^{l_E}. Then*

$$\dim\widetilde{\mathbf{R}} = \sum_{\left\langle v_i\in V(G^{l_E})|1\leq i\leq s\right\rangle\in CL_s(G^{l_E})} (-1)^{s+1}\dim\left(\mathbf{R}^{n_{v_1}}\bigcap\mathbf{R}^{n_{v_2}}\bigcap\cdots\bigcap\mathbf{R}^{n_{v_s}}\right),$$

where n_{v_i} denotes the dimensional number of the Euclidean space in $v_i\in V(G^{l_E})$ and $CL_s(G^{l_E})$ consists of all complete subgraphs of order s in G^{l_E}.

Proof By definition, $\mathbf{R}^{n_u}\cap\mathbf{R}^{n_v}\neq\emptyset$ only if there is an edge $(\mathbf{R}^{n_u},\mathbf{R}^{n_v})$ in G^{l_E}, which can be generalized to a more general situation, i.e., $\mathbf{R}^{n_{v_1}}\cap\mathbf{R}^{n_{v_2}}\cap\cdots\cap\mathbf{R}^{n_{v_l}}\neq\emptyset$ only if $\langle v_1,v_2,\cdots,v_l\rangle_{G^{l_E}}\simeq K_l$. In fact, if $\mathbf{R}^{n_{v_1}}\cap\mathbf{R}^{n_{v_2}}\cap\cdots\cap\mathbf{R}^{n_{v_l}}\neq\emptyset$, then $\mathbf{R}^{n_{v_i}}\cap\mathbf{R}^{n_{v_j}}\neq\emptyset$, which implies that $(\mathbf{R}^{n_{v_i}},\mathbf{R}^{n_{v_j}})\in E(G^{l_E})$ for any integers $i,j,\ 1\leq i,j\leq l$. Thus $\langle v_1,v_2,\cdots,v_l\rangle_{G^{l_E}}$ is a complete subgraph of order l in graph G^{l_E}.

Notice that the number of different orthogonal elements is $\dim\widetilde{\mathbf{R}}=\dim\left(\bigcup\limits_{v\in V(G^{l_E})}\mathbf{R}^{n_v}\right)$. Applying the inclusion-exclusion principle, we get that

$$\begin{aligned}\dim\widetilde{\mathbf{R}} &= \dim\left(\bigcup_{v\in V(G^{l_E})}\mathbf{R}^{n_v}\right)\\ &= \sum_{\{v_1,\cdots,v_s\}\subset V(G^{l_E})}(-1)^{s+1}\dim\left(\mathbf{R}^{n_{v_1}}\bigcap\mathbf{R}^{n_{v_2}}\bigcap\cdots\bigcap\mathbf{R}^{n_{v_s}}\right)\\ &= \sum_{\left\langle v_i\in V(G^{l_E})|1\leq i\leq s\right\rangle\in CL_s(G^{l_E})}(-1)^{s+1}\dim\left(\mathbf{R}^{n_{v_1}}\bigcap\mathbf{R}^{n_{v_2}}\bigcap\cdots\bigcap\mathbf{R}^{n_{v_s}}\right). \qquad\square\end{aligned}$$

Notice that $\dim(\mathbf{R}^{n_{v_1}}\cap\mathbf{R}^{n_{v_2}})\neq 0$ only if $(\mathbf{R}^{n_{v_1}},\mathbf{R}^{n_{v_2}})\in E(G^{l_E})$. We get an applicable formula for $\dim\widetilde{\mathbf{R}}$ on K_3-free graphs G^{l_E}, i.e., there are no subgraphs of G^{l_E} isomorphic to K_3 by Theorem 9.2.3 following.

Corollary 9.2.2 *Let $\widetilde{\mathbf{R}}$ be a combinatorial Euclidean space underlying a K_3-free edge-labeled graph G^{l_E}. Then*

$$\dim\widetilde{\mathbf{R}} = \sum_{v\in V(G^{l_E})} n_v - \sum_{(u,v)\in E(G^{l_E})}\dim\left(\mathbf{R}^{n_u}\bigcap\mathbf{R}^{n_v}\right).$$

Particularly, if $G=v_1v_2\cdots v_m$ a circuit for an integer $m\geq 4$, then

$$\dim\widetilde{\mathbf{R}} = \sum_{i=1}^{m} n_{v_i} - \sum_{i=1}^{m}\dim\left(\mathbf{R}^{n_{v_i}}\bigcap\mathbf{R}^{n_{v_{i+1}}}\right),$$

where each index is modulo m.

9.2.3 Decomposition Space into Combinatorial One. A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is a combinatorial Euclidean space $\widetilde{\mathbf{R}}$ consists of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers $i, j,\ 1 \le i \neq j \le m$,

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$

The dimensional number of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is

$$\dim \widetilde{\mathbf{R}}\,(n_1, \cdots, n_m) = \widehat{m} + \sum_{i=1}^{m} (n_i - \widehat{m}),$$

determined immediately by definition, where $\widehat{m} = \dim\left(\bigcap\limits_{k=1}^{m} \mathbf{R}^{n_k}\right)$.

For visualizing the WORLD, *weather is there a combinatorial Euclidean space, particularly, a combinatorial fan-space* $\widetilde{\mathbf{R}}$ *consisting of Euclidean spaces* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *for a Euclidean space* $\mathbf{R}^n$ *such that* $\dim \mathbf{R}^{n_1} \cup \mathbf{R}^{n_2} \cup \cdots \cup \mathbf{R}^{n_m} = n$? We know the following decomposition result of Euclidean spaces.

Theorem 9.2.4 *Let* $\mathbf{R}^n$ *be a Euclidean space,* $n_1, n_2, \cdots, n_m$ *integers with* $\widehat{m} < n_i < n$ *for* $1 \le i \le m$ *and the equation*

$$\widehat{m} + \sum_{i=1}^{m} (n_i - \widehat{m}) = n$$

holds for an integer $\widehat{m}, 1 \le \widehat{m} \le n$. *Then there is a combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ *such that*

$$\mathbf{R}^n \simeq \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m).$$

Proof Not loss of generality, we assume the normal basis of $\mathbf{R}^n$ is $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$. Since

$$n - \widehat{m} = \sum_{i=1}^{m} (n_i - \widehat{m}),$$

choose

$$\mathbf{R}_1 = \langle \overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_{\widehat{m}}, \overline{\epsilon}_{\widehat{m}+1}, \cdots, \overline{\epsilon}_{n_1} \rangle;$$
$$\mathbf{R}_2 = \langle \overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_{\widehat{m}}, \overline{\epsilon}_{n_1+1}, \overline{\epsilon}_{n_1+2}, \cdots, \overline{\epsilon}_{n_2} \rangle;$$
$$\mathbf{R}_3 = \langle \overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_{\widehat{m}}, \overline{\epsilon}_{n_2+1}, \overline{\epsilon}_{n_2+2}, \cdots, \overline{\epsilon}_{n_3} \rangle;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$\mathbf{R}_m = \left\langle \overline{\epsilon}_1, \overline{\epsilon}_2, \cdots, \overline{\epsilon}_{\widehat{m}}, \overline{\epsilon}_{n_{m-1}+1}, \overline{\epsilon}_{n_{m-1}+2}, \cdots, \overline{\epsilon}_{n_m} \right\rangle.$$

Calculation shows that $\dim\mathbf{R}_i = n_i$ and $\dim(\bigcap\limits_{i=1}^{m} \mathbf{R}_i) = \widehat{m}$. Whence $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is a combinatorial fan-space. Thus

$$\mathbf{R}^n \simeq \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m). \qquad \square$$

Corollary 9.2.3 *For a Euclidean space $\mathbf{R}^n$, there is a combinatorial Euclidean fan-space $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ underlying a complete graph K_m with $\widehat{m} < n_i < n$ for integers $1 \leq i \leq m$, $\widehat{m} + \sum\limits_{i=1}^{m} (n_i - \widehat{m}) = n$ such that $\mathbf{R}^n \simeq \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$.*

§9.3 COMBINATORIAL MANIFOLDS

9.3.1 Combinatorial Manifold. For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a *combinatorial manifold* $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$$

and

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\},$$

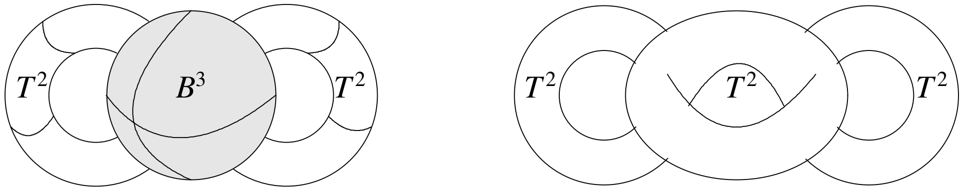
denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context, and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p) = \dim\left(\bigcap\limits_{i=1}^{s(p)} \mathbf{R}^{n_i(p)}\right)$ are called the *dimension* and the *intersectional dimension* of $\widetilde{M}(n_1, \cdots, n_m)$ at the point p, respectively.

A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. Certainly, a finitely combinatorial manifold is indeed a combinatorial manifold.

Two examples of such combinatorial manifolds with different dimensions in $\mathbf{R}^3$ are shown in Fig.9.3.1, in where, (a) represents a combination of a 3-manifold with two tori, and (b) three tori.



(a) (b)

Fig.9.3.1

By definition, combinatorial manifolds are nothing but a generalization of manifolds by combinatorial speculation. However, a locally compact n-manifold M^n without boundary is itself a combinatorial Euclidean space $\widetilde{\mathbf{R}}(n)$ of Euclidean spaces $\mathbf{R}^n$ with an underlying structure G^{l_E} shown in the next result.

Theorem 9.3.1 *A locally compact n-manifold* M^n *is a combinatorial manifold* $\widetilde{M}_G(n)$ *homeomorphic to a Euclidean space* $\widetilde{\mathbf{R}}(n, \lambda \in \Lambda)$ *with countable graphs* G^{l_E} *inherent in* M^n*, denoted by* $G[M^n]$.

Proof Let M^n be a locally compact n-manifold with an atlas

$$\mathscr{A}[M^n] = \{ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\},$$

where Λ is a countable set. Then each U_λ, $\lambda \in \Lambda$ is itself an n-manifold by definition. Define an underlying combinatorial structure G^{l_E} by

$$V(G^{l_E}) = \{U_\lambda | \lambda \in \Lambda\},$$

$$E(G^{l_E}) = \{ (U_\lambda, U_\iota)_i, 1 \leq i \leq \kappa_{\lambda\iota} + 1 | \ U_\lambda \bigcap U_\iota \neq \emptyset, \lambda, \iota \in \Lambda\}$$

where $\kappa_{\lambda\iota}$ is the number of non-homotopic loops formed between U_λ and U_ι. Then we get a combinatorial manifold M^n underlying a countable graph G^{l_E}.

Define a combinatorial Euclidean space $\widetilde{\mathbf{R}}(n, \lambda \in \Lambda)$ of Euclidean spaces $\mathbf{R}^n$ by

$$V(G^{l_E}) = \{\varphi_\lambda(U_\lambda) | \lambda \in \Lambda\},$$

$$E(G^{l_E}) = \{ (\varphi_\lambda(U_\lambda), \varphi_\iota(U_\iota))_i, 1 \leq i \leq \kappa'_{\lambda\iota} + 1 | \ \varphi_\lambda(U_\lambda) \bigcap \varphi_\iota(U_\iota) \neq \emptyset, \lambda, \iota \in \Lambda\},$$

where $\kappa'_{\lambda\iota}$ is the number of non-homotopic loops in formed between $\varphi_\lambda(U_\lambda)$ and $\varphi_\iota(U_\iota)$. Notice that $\varphi_\lambda(U_\lambda)\bigcap\varphi_\iota(U_\iota)\neq\emptyset$ if and only if $U_\lambda\bigcap U_\iota\neq\emptyset$ and $\kappa_{\lambda\iota}=\kappa'_{\lambda\iota}$ for $\lambda,\iota\in\Lambda$.

Now we prove that M^n is homeomorphic to $\widetilde{\mathbf{R}}(n,\lambda\in\Lambda)$. By assumption, M^n is locally compact. Whence, there exists a partition of unity $c_\lambda: U_\lambda\rightarrow\mathbf{R}^n$, $\lambda\in\Lambda$ on the atlas $\mathscr{A}[M^n]$. Let $A_\lambda=\operatorname{supp}(\varphi_\lambda)$. Define functions $h_\lambda: M^n\rightarrow\mathbf{R}^n$ and $\mathbf{H}: M^n\rightarrow\mathscr{E}_{G'}(n)$ by

$$h_\lambda(x)=\begin{cases} c_\lambda(x)\varphi_\lambda(x) & \text{if } x\in U_\lambda,\\ \mathbf{0}=(0,\cdots,0) & \text{if } x\in U_\lambda-A_\lambda.\end{cases}$$

and

$$\mathbf{H}=\sum_{\lambda\in\Lambda}\varphi_\lambda c_\lambda, \quad \text{and} \quad \mathbf{J}=\sum_{\lambda\in\Lambda}c_\lambda^{-1}\varphi_\lambda^{-1}.$$

Then h_λ, $\mathbf{H}$ and $\mathbf{J}$ all are continuous by the continuity of φ_λ and c_λ for $\forall\lambda\in\Lambda$ on M^n. Notice that $c_\lambda^{-1}\varphi_\lambda^{-1}\varphi_\lambda c_\lambda$ =the unity function on M^n. We get that $\mathbf{J}=\mathbf{H}^{-1}$, i.e., $\mathbf{H}$ is a homeomorphism from M^n to $\mathscr{E}_{G'}(n,\lambda\in\Lambda)$. □

9.3.2 Combinatorial *d*-Connected Manifold. For two points p,q in a finitely combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$, if there is a sequence $B_1,B_2,\cdots,B_s$ of d-dimensional open balls with two conditions following hold:

(1) $B_i\subset\widetilde{M}(n_1,n_2,\cdots,n_m)$ for any integer i, $1\leq i\leq s$ and $p\in B_1$, $q\in B_s$;

(2) The dimensional number $\dim(B_i\bigcap B_{i+1})\geq d$ for $\forall i$, $1\leq i\leq s-1$,

then points p,q are called d-dimensional connected in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ and the sequence $B_1,B_2,\cdots,B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p,q)$. If each pair p,q of points in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is called d-pathwise connected and say its connectivity$\geq d$.

Not loss of generality, we consider only finitely combinatorial manifolds with a connectivity≥ 1 in this book. Let $\widetilde{M}(n_1,n_2,\cdots,n_m)$ be a finitely combinatorial manifold and $d\geq 1$ an integer. We construct a vertex-edge labeled graph $G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)]$ by

$$V(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=V_1\bigcup V_2,$$

where,

$V_1=\{n_i-\text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1,n_2,\cdots,n_m)|1\leq i\leq m\}$ and

$V_2=\{\text{isolated intersection points } O_{M^{n_i},M^{n_j}} \text{ of} M^{n_i},M^{n_j} \text{ in } \widetilde{M}(n_1,n_2,\cdots,n_m) \text{ for } 1\leq i,j\leq m\}$. Label n_i for each n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1,n_2,\cdots,n_m)])=E_1\bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j})$ labeled with $\dim(M^{n_i} \bigcap M^{n_j}) \mid \dim(M^{n_i} \bigcap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$ and $E_2 = \{(O_{M^{n_i},M^{n_j}}, M^{n_i}), (O_{M^{n_i},M^{n_j}}, M^{n_j})$ labeled with $0 \mid M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i},M^{n_j}}$ for $1 \leq i, j \leq m\}$.

For example, these correspondent labeled graphs gotten from finitely combinatorial manifolds in Fig.9.3.1 are shown in Fig.9.3.2, in where $d = 2$ for (a) and (b). Notice if $\dim(M^{n_i} \cap M^{n_j}) \leq d - 1$, then there are no such edges (M^{n_i}, M^{n_j}) in $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$.

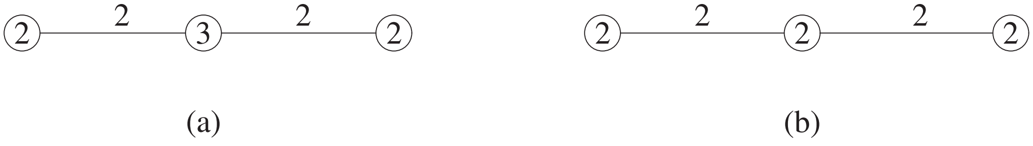


Fig.9.3.2

Theorem 9.3.2 *Let $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ be a labeled graph of a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$. Then*

(1) *$G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is connected only if $d \leq n_1$.*

(2) *there exists an integer $d, d \leq n_1$ such that $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is connected.*

Proof By definition, there is an edge (M^{n_i}, M^{n_j}) in $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ for $1 \leq i, j \leq m$ if and only if there is a d-dimensional path $P^d(p, q)$ connecting two points $p \in M^{n_i}$ and $q \in M^{n_j}$. Notice that

$$(P^d(p, q) \setminus M^{n_i}) \subseteq M^{n_j} \text{ and } (P^d(p, q) \setminus M^{n_j}) \subseteq M^{n_i}.$$

Whence,

$$d \leq \min\{n_i, n_j\}. \qquad (9-3-1)$$

Now if $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is connected, then there is a d-path $P(M^{n_i}, M^{n_j})$ connecting vertices M^{n_i} and M^{n_j} for $\forall M^{n_i}, M^{n_j} \in V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$. Not loss of generality, assume

$$P(M^{n_i}, M^{n_j}) = M^{n_i} M^{s_1} M^{s_2} \cdots M^{s_{t-1}} M^{n_j}.$$

Then we get that

$$d \leq \min\{n_i, s_1, s_2, \cdots, s_{t-1}, n_j\} \qquad (9-3-2)$$

by $(9-3-1)$. However, by definition we know that

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}. \qquad (9-3-3)$$

Therefore, we get that

$$d \leq \min\left(\bigcup_{p\in\widetilde{M}}\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\}\right) = \min\{n_1, n_2, \cdots, n_m\} = n_1$$

by combining $(9-3-2)$ with $(9-3-3)$. Notice that points labeled with 0 and 1 are always connected by a path. We get the conclusion (1).

For the conclusion (2), notice that any finitely combinatorial manifold is always pathwise 1-connected by definition. Accordingly, $G^1[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ is connected. Thereby, there at least one integer, for instance $d = 1$ enabling $G^d\left[\widetilde{M}(n_1, n_2, \cdots, n_m)\right]$ to be connected. This completes the proof. □

According to Theorem 9.3.2, we get immediately two conclusions following.

Corollary 9.3.1 *For a given finitely combinatorial manifold $\widetilde{M}$, all connected graphs $G^d\left[\widetilde{M}\right]$ are isomorphic if $d \leq n_1$, denoted by $G^L\left[\widetilde{M}\right]$.*

Corollary 9.3.2 *If there are k 1-manifolds intersect at one point p in a finitely combinatorial manifold $\widetilde{M}$, then there is an induced subgraph K^{k+1} in $G^L\left[\widetilde{M}\right]$.*

Now we define an edge set $E^d\left(\widetilde{M}\right)$ in $G^L\left[\widetilde{M}\right]$ by

$$E^d\left(\widetilde{M}\right) = E\left(G^d\left[\widetilde{M}\right]\right) \setminus E\left(G^{d+1}\left[\widetilde{M}\right]\right).$$

Then we get a graphical recursion equation for graphs of a finitely combinatorial manifold $\widetilde{M}$ as a by-product.

Theorem 9.3.3 *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then for any integer $d, d \geq 1$, there is a recursion equation $G^{d+1}\left[\widetilde{M}\right] = G^d\left[\widetilde{M}\right] - E^d\left(\widetilde{M}\right)$ for labeled graphs of $\widetilde{M}$.*

Proof It can be obtained immediately by definition. □

Now let $\mathcal{H}(n_1, \cdots, n_m)$ denotes all finitely combinatorial manifolds $\widetilde{M}(n_1, \cdots, n_m)$ and $\mathcal{G}[0, n_m]$ all vertex-edge labeled graphs G^L with $\theta_L : V(G^L) \cup E(G^L) \to \{0, 1, \cdots, n_m\}$ with conditions following hold.

(1)Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2)For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0, n_m], [0, n_m])$ following.

Theorem 9.3.4 *Let* $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ *be a given integer sequence. Then every finitely combinatorial manifold* $\widetilde{M} \in \mathcal{H}(n_1, \cdots, n_m)$ *defines a vertex-edge labeled graph* $G([0, n_m]) \in \mathcal{G}[0, n_m]$. *Conversely, every vertex-edge labeled graph* $G([0, n_m]) \in \mathcal{G}[0, n_m]$ *defines a finitely combinatorial manifold* $\widetilde{M} \in \mathcal{H}(n_1, \cdots, n_m)$ *with a* $1-1$ *mapping* $\theta : G([0, n_m]) \to \widetilde{M}$ *such that* $\theta(u)$ *is a* $\theta(u)$*-manifold in* $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ *and* $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ *for* $\forall u \in V(G([0, n_m]))$ *and* $\forall(v, w) \in E(G([0, n_m]))$.

Proof By definition, for $\forall\widetilde{M} \in \mathcal{H}(n_1, \cdots, n_m)$ there is a vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}([0, n_m])$ and a $1-1$ mapping $\theta : \widetilde{M} \to G([0, n_m])$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$. For completing the proof, we need to construct a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, \cdots, n_m)$ for $\forall G([0, n_m]) \in \mathcal{G}[0, n_m]$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall(v, w) \in E(G([0, n_m]))$. The construction is carried out by programming following.

STEP 1. Choose $|G([0, n_m])| - |V_0|$ manifolds correspondent to each vertex u with a dimensional n_i if $\tau_1(u) = n_i$, where $V_0 = \{u | u \in V(G([0, n_m]))$ and $\tau_1(u) = 0\}$. Denoted by $V_{\geq 1}$ all these vertices in $G([0, n_m])$ with label≥ 1.

STEP 2. For $\forall u_1 \in V_{\geq 1}$ with $\tau_1(u_1) = n_{i_1}$, if its neighborhood set $N_{G([0,n_m])}(u_1) \bigcap V_{\geq 1} = \{v_1^1, v_1^2, \cdots, v_1^{s(u_1)}\}$ with $\tau_1(v_1^1) = n_{11}$, $\tau_1(v_1^2) = n_{12}$, $\cdots$, $\tau_1(v_1^{s(u_1)}) = n_{1s(u_1)}$, then let the manifold correspondent to the vertex u_1 with an intersection dimension $\tau_2(u_1 v_1^i)$ with manifold correspondent to the vertex v_1^i for $1 \leq i \leq s(u_1)$ and define a vertex set $\Delta_1 = \{u_1\}$.

STEP 3. If the vertex set $\Delta_l = \{u_1, u_2, \cdots, u_l\} \subseteq V_{\geq 1}$ has been defined and $V_{\geq 1} \setminus \Delta_l \neq \emptyset$, let $u_{l+1} \in V_{\geq 1} \setminus \Delta_l$ with a label $n_{i_{l+1}}$. Assume

$$(N_{G([0,n_m])}(u_{l+1}) \bigcap V_{\geq 1}) \setminus \Delta_l = \{v_{l+1}^1, v_{l+1}^2, \cdots, v_{l+1}^{s(u_{l+1})}\}$$

with $\tau_1(v_{l+1}^1) = n_{l+1,1}$, $\tau_1(v_{l+1}^2) = n_{l+1,2}$, $\cdots$,$\tau_1(v_{l+1}^{s(u_{l+1})}) = n_{l+1,s(u_{l+1})}$. Then let the manifold correspondent to the vertex u_{l+1} with an intersection dimension $\tau_2(u_{l+1} v_{l+1}^i)$ with the manifold correspondent to the vertex $v_{l+1}^i, 1 \leq i \leq s(u_{l+1})$ and define a vertex set $\Delta_{l+1} = \Delta_l \bigcup \{u_{l+1}\}$.

STEP 4. Repeat steps 2 and 3 until a vertex set $\Delta_t = V_{\geq 1}$ has been constructed. This construction is ended if there are no vertices $w \in V(G)$ with $\tau_1(w) = 0$, i.e., $V_{\geq 1} = V(G)$. Otherwise, go to the next step.

STEP 5. For $\forall w \in V(G([0, n_m])) \setminus V_{\geq 1}$, assume $N_{G([0,n_m])}(w) = \{w_1, w_2, \cdots, w_e\}$. Let all these manifolds correspondent to vertices $w_1, w_2, \cdots, w_e$ intersects at one point simultaneously and define a vertex set $\Delta^*_{t+1} = \Delta_t \bigcup \{w\}$.

STEP 6. Repeat STEP 5 for vertices in $V(G([0, n_m])) \setminus V_{\geq 1}$. This construction is finally ended until a vertex set $\Delta^*_{t+h} = V(G[n_1, n_2, \cdots, n_m])$ has been constructed.

A finitely combinatorial manifold $\widetilde{M}$ correspondent to $G([0, n_m])$ is gotten when Δ^*_{t+h} has been constructed. By this construction, it is easily verified that $\widetilde{M} \in \mathcal{H}(n_1, \cdots, n_m)$ with $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m]))$. This completes the proof. $\square$

9.3.3 Euler-Poincaré Characteristic. the Euler-Poincaré characteristic of a CW-complex $\mathfrak{M}$ is defined to be the integer

$$\chi(\mathfrak{M}) = \sum_{i=0}^{\infty} (-1)^i \alpha_i$$

with α_i the number of i-dimensional cells in $\mathfrak{M}$. We calculate the Euler-Poincaré characteristic of finitely combinatorial manifolds in this subsection. For this objective, define a clique sequence $\{Cl(i)\}_{i\geq 1}$ in the graph $G^L\left[\widetilde{M}\right]$ by the following programming.

STEP 1. Let $Cl\left(G^L\left[\widetilde{M}\right]\right) = l_0$. Construct

$$\begin{aligned} Cl(l_0) \;=\; & \Big\{K_1^{l_0}, K_2^{l_0}, \cdots, K_p^{i_0} | K_i^{l_0} \succ G^L[\widetilde{M}] \text{ and } K_i^{l_0} \cap K_j^{l_0} = \emptyset, \\ & \text{or a vertex} \in \mathrm{V}\left(\mathrm{G^L}\left[\widetilde{\mathrm{M}}\right]\right) \text{ for } i \neq j, 1 \leq i, j \leq p\Big\}. \end{aligned}$$

STEP 2. Let $G_1 = \bigcup\limits_{K^l \in Cl(l)} K^l$ and $Cl\left(G^L\left[\widetilde{M}\right] \setminus G_1\right) = l_1$. Construct

$$\begin{aligned} Cl(l_1) \;=\; & \Big\{K_1^{l_1}, K_2^{l_1}, \cdots, K_q^{i_1} | K_i^{l_1} \succ G^L[\widetilde{M}] \text{ and } K_i^{l_1} \cap K_j^{l_1} = \emptyset \\ & \text{or a vertex} \in \mathrm{V}\left(\mathrm{G^L}\left[\widetilde{\mathrm{M}}\right]\right) \text{ for } i \neq j, 1 \leq i, j \leq q\Big\}. \end{aligned}$$

STEP 3. Assume we have constructed $Cl(l_{k-1})$ for an integer $k \geq 1$. Let $G_k = \bigcup\limits_{K^{l_{k-1}} \in Cl(l)} K^{l_{k-1}}$ and $Cl\left(G^L\left[\widetilde{M}\right] \setminus (G_1 \cup \cdots \cup G_k)\right) = l_k$. We construct

$$\begin{aligned} Cl(l_k) \;=\; & \Big\{K_1^{l_k}, K_2^{l_k}, \cdots, K_r^{l_k} | K_i^{l_k} \succ G^L[\widetilde{M}] \text{ and } K_i^{l_k} \cap K_j^{l_k} = \emptyset, \\ & \text{or a vertex} \in \mathrm{V}\left(\mathrm{G^L}\left[\widetilde{\mathrm{M}}\right]\right) \text{ for } i \neq j, 1 \leq i, j \leq r\Big\}. \end{aligned}$$

STEP 4. Continue STEP 3 until we find an integer t such that there are no edges in $G^L\left[\widetilde{M}\right] \setminus \bigcup\limits_{i=1}^{t} G_i$.

By this clique sequence $\{Cl(i)\}_{i\geq 1}$, we can calculate the Euler-Poincaré characteristic of finitely combinatorial manifolds.

Theorem 9.3.5 *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then*

$$\chi\left(\widetilde{M}\right) = \sum_{K^k\in Cl(k),k\geq 2}\ \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\chi\left(M_{i_1}\bigcap\cdots\bigcap M_{i_s}\right)$$

Proof Denoted the numbers of all these i-dimensional cells in a combinatorial manifold $\widetilde{M}$ or in a manifold M by $\widetilde{\alpha}_i$ and $\alpha_i(M)$. If $G^L\left[\widetilde{M}\right]$ is nothing but a complete graph K^k with $V\left(G^L\left[\widetilde{M}\right]\right) = \{M_1, M_2, \cdots, M_k\}$, $k \geq 2$, by applying the inclusion-exclusion principe and the definition of Euler-Poincaré characteristic we get that

$$\begin{aligned}
\chi\left(\widetilde{M}\right) &= \sum_{i=0}^{\infty}(-1)^i\widetilde{\alpha}_i \\
&= \sum_{i=0}^{\infty}(-1)^i \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\alpha_i\left(M_{i_1}\bigcap\cdots\bigcap M_{i_s}\right) \\
&= \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\sum_{i=0}^{\infty}(-1)^i\alpha_i\left(M_{i_1}\bigcap\cdots\bigcap M_{i_s}\right) \\
&= \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{s+1}\chi\left(M_{i_1}\bigcap\cdots\bigcap M_{i_s}\right)
\end{aligned}$$

for instance, $\chi\left(\widetilde{M}\right) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2)$ if $G^L\left[\widetilde{M}\right] = K^2$ and $V\left(G^L\left[\widetilde{M}\right]\right) = \{M_1, M_2\}$. By the definition of clique sequence of $G^L\left[\widetilde{M}\right]$, we finally obtain that

$$\chi\left(\widetilde{M}\right) = \sum_{K^k\in Cl(k),k\geq 2}\ \sum_{M_{i_j}\in V(K^k),1\leq j\leq s\leq k} (-1)^{i+1}\chi\left(M_{i_1}\bigcap\cdots\bigcap M_{i_s}\right). \qquad \square$$

Particularly, if $G^L\left[\widetilde{M}\right]$ is one of some special graphs, we can get interesting consequences by Theorem 9.3.5.

Corollary 9.3.3 *Let $\widetilde{M}$ be a finitely combinatorial manifold. If $G^L\left[\widetilde{M}\right]$ is K^3-free, then*

$$\chi\left(\widetilde{M}\right) = \sum_{M\in V(G^L[\widetilde{M}])} \chi^2(M) - \sum_{(M_1,M_2)\in E(G^L[\widetilde{M}])} \chi\left(M_1\bigcap M_2\right).$$

Particularly, if $\dim\left(M_1 \bigcap M_2\right)$ *is a constant for any* $(M_1, M_2) \in E\left(G^L\left[\widetilde{M}\right]\right)$, *then*

$$\chi\left(\widetilde{M}\right) = \sum_{M\in V(G^L[\widetilde{M}])} \chi^2(M) - \chi\left(M_1\bigcap M_2\right)\left|E\left(G^L\left[\widetilde{M}\right]\right)\right|.$$

Proof Notice that $G^L[\widetilde{M}]$ is K^3-free, we get that

$$\begin{aligned}\chi\left(\widetilde{M}\right) &= \sum_{(M_1,M_2)\in E(G^L[\widetilde{M}])} \left(\chi(M_1)+\chi(M_2)-\chi\left(M_1\bigcap M_2\right)\right)\\ &= \sum_{(M_1,M_2)\in E(G^L[\widetilde{M}])} (\chi(M_1)+\chi(M_2)) - \sum_{(M_1,M_2)\in E(G^L[\widetilde{M}])} \chi\left(M_1\bigcap M_2\right)\\ &= \sum_{M\in V(G^L[\widetilde{M}])} \chi^2(M) - \sum_{(M_1,M_2)\in E(G^L[\widetilde{M}])} \chi\left(M_1\bigcap M_2\right).\end{aligned}$$

□

Since the Euler-Poincaré characteristic of a manifold M is 0 if $\dim M \equiv 1(mod 2)$, we get the following consequence.

Corollary 9.3.4 *Let $\widetilde{M}$ be a finitely combinatorial manifold with odd dimension number for any intersection of k manifolds with $k \geq 2$. Then*

$$\chi\left(\widetilde{M}\right) = \sum_{M\in V(G^L[\widetilde{M}])} \chi(M).$$

§9.4 TOPOLOGICAL SPACES COMBINING WITH MULTI-GROUPS

9.4.1 Topological Group. A topological group is a combination of topological space with that of group formally defined following.

Definition 9.4.1 *A topological group $(G;\circ)$ is a Hausdorff topological space G together with a group structure on $(G;\circ)$ satisfying conditions following:*

(1) *The group multiplication $\circ: (a,b) \to a\circ b$ of $G\times G \to G$ is continuous;*

(2) *The group inversion $g \to g^{-1}$ of $G \to G$ is continuous.*

Notice that these conditions (1) and (2) can be restated following by the definition of continuous mapping.

(1′) *Let $a,b \in G$. Then for any neighborhood W of $a\circ b$, there are neighborhoods U, V of a and b such that $UV \subset W$, where $UV = \{x\circ y | x \in U,\ y \in V\}$;*

(2′) *For $a \in G$ and any neighborhood V of a^{-1}, there is a neighborhood U of a such that $U \subset V$.*

It is easily verified that conditions (1) and (2) can be replaced by a condition (3) following:

(3′) *Let $a, b \in G$. Then for any neighborhood W of $a \circ b^{-1}$, there are neighborhoods U, V of a and b such that $UV^{-1} \subset W$.*

A few examples of topological group are listed in the following.

(1) $(\mathbb{R}^n; +)$ and $(\mathbb{C}^n; +)$, the additive groups of n-tuple of real or complex numbers are topological group.

(2) The multiplicative group $(\mathbb{S}; \cdot)$ of the complex numbers with $\mathbb{S} = \{\, z \mid |z| = 1\}$ is a topological group with structure $\mathbf{S}^1$.

(3) Let $(\mathbb{C}^*; \cdot)$ be the multiplicative group of non-zero complex numbers. The topological structure of $(\mathbb{C}^*; \cdot)$ is $\mathbb{R}^2 - \{(0, 0)\}$, an open submanifold of complex plane $\mathbb{R}^2$. Whence, it is a topological group.

(4) Let $Gl(n, \mathbb{R})$ be the set of $n \times n$ non-singular matrices M_n, which is a Euclidean space of $\mathbb{R}^n - (\underbrace{0, 0, \cdots, 0}_{n})$. Notice that the determinant function det : $M_n \to \mathbb{R}$ is continuous because it is nothing but a polynomial in the coefficients of M_n. Thus $(Gl(n, \mathbb{R}); \det)$ is a topological group.

Some elementary properties of a topological group $(G; \circ)$ are listed following.

(**P1**) Let $a_i \in G$ for integers $1 \leq i \leq n$ and $a_1^{\epsilon_1} \circ a_2^{\epsilon_1} \circ \cdots \circ a_n^{\epsilon_n} = b$, where ϵ_i is an integer. By condition (1′), for a $V(b)$ neighborhood of b, there exist neighborhoods $U_1, U_2, \cdots, U_n$ of $a_1, a_2, \cdots, a_n$ such that $U_1^{\epsilon_1} \circ U_2^{\epsilon_1} \circ \cdots \circ U_n^{\epsilon_n} \subset V(b)$.

(**P2**) Let $a \in G$ be a chosen element and $f(x) = x \circ a$, $f'(x) = a \circ x$ and $\phi(x) = x^{-1}$ for $\forall x \in G$. It is clear that f, f' and ϕ are bijection on G. They are also continuous. In fact, let $b' = x' \cdot a$ for $x' \in G$ and V a neighborhood of b'. By condition (1′), there are neighborhoods U, W of x' and a such that $UW \subset V$. Notice that $a \in W$. Thus $Ua \subset V$. By definition, we know that f is continuous. Similarly, we know that f' and ϕ are continuous. Whence, f, f' and ϕ are homeomorphism on G.

(**P3**) Let $E, F \subset$ be open and closed subsets, respectively. Then for $\forall a \in G$, by property (P2), Ea, aE, E^{-1} are open, and Fa, aF, F^{-1} are closed also.

(**P4**) A topological space S is *homogenous* if there is a homeomorphism $T : S \to S$ for $\forall p, q \in S$ such that $T(p) = q$. Let $a = p^{-1} \circ q$ in (P2). We know immediately that the topological group $(G; \circ)$ is homogenous.

9.4.2 Topological Subgroup. Let $(G;\circ)$ be a topological group and $H\subset G$ with conditions following hold:

(1) $(H,\circ)$ is a subgroup of $(G;\circ)$;

(2) H is closed.

Such a subgroup $(H;\circ)$ is called a *topological subgroup* of $(G;\circ)$.

Theorem 9.4.1 *Let* $(G;\circ)$ *be a topological group and let* $(H;\circ)$ *be an algebraic subgroup of* $(G;\circ)$. *Then* $(H;\circ)$ *is a topological subgroup of* $(G;\circ)$ *with an induced topology, i.e., S is open if and only if $S = H\bigcap T$, where T is open in G. Furthermore,* $(\overline{H};\ irc)$ *is a topological subgroup of* $(G;\circ)$ *and if $H\lhd G$, then* $(\overline{H};\circ)$ *is a topological normal subgroup of* $(G;\circ)$.

Proof We only need to prove that $\circ : H\times H\to H$ is continuous. Let $a,b\in H$ with $a\circ b^{-1}=c$ and W a neighborhood of c in H. Then there is an open neighborhood W' of c in G such that $W=H\bigcap W'$. Since $(G;\circ)$ is a topological group, there are neighborhoods U', V' of a and b respectively such that $U'(V')^{-1}\subset W'$. Notice that $U=H\bigcap U'$ and $V=H\bigcap V'$ are neighborhoods of a and b in H by definition. We know that $UV^{-1}\subset W$. Thus $(H;\circ)$ is a topological subgroup.

Now let $a,b\in\overline{H}$. Then $a\circ b^{-1}\in\overline{H}$. In fact, by $a,b\in\overline{H}$, there exist elements $x,y\in H$ such that $x\circ y^{-1}\in H\bigcap W$, which implies that $a\circ b^{-1}\in\overline{H}$. Whence $(\overline{H};\ irc)$ is a topological subgroup.

For $\forall c\in G$ and $a\in H$, if $H\lhd G$, then $c\circ a\circ c^{-1}\in H$. Let V be a neighborhood of $c\circ a\circ c^{-1}$. Then there is a neighborhood of U such that $cUc^{-1}\subset V$. Since $a\in\overline{H}$, there exist $x\in H\bigcap U$ such that $c\circ x\circ c^{-1}\in H\bigcap V$ Thus $c\circ x\circ c^{-1}\in\overline{H}$. Whence, $(\overline{H};\circ)$ is a topological normal subgroup of $(G;\circ)$. □

Similarly, there are two topological normal subgroups in any topological group $(G;\circ)$, i.e., $\{1_G\}$ and $(G;\circ)$ itself. A topological group only has topological normal subgroups $\{1_G\}$ and $(G;\circ)$ is called a *simple topological group*.

9.4.3 Quotient Topological Group. Let $(G;\circ)$ be a topological group and let $H\lhd G$ be a normal subgroup of $(G;\circ)$. Consider the quotient G/H with the quotient topology, namely the finest topology on G/H that makes the canonical projection $q : G\to G/H$ continuous. Such a quotient topology consists of all sets $q(U)$, where U runs over the family of all open sets of $(G;\circ)$. Whence, if $U\subset G$ is open, then $q^{-1}q(U)=UH=\{Uh|h\in H\}$ is the union of open sets, and so it is also open.

Choose $A, B \in G/H$, $a \in A$. Notice that $B = bH$ is closed and $a \notin B$. There exists a neighborhood $U(a)$ of a with $U(a) \bigcap B = \emptyset$. Thus the set U' consisting of all $xH,\ x \in U$ is a neighborhood of A with $U' \bigcap B = \emptyset$. Thus $(G/H; \circ)$ is Hausdorff space.

By definition, $q(a) = aH$. Let U' be a neighborhood of a, i.e., consisting of xH with $x \in U$ and U a neighborhood of $(G; \circ)$. Notice that UH is open and $a \in UH$. Whence, there is a neighborhood V of a such that $V \subset UH$. Clearly, $q(V) \subset U'$. Thus q is continuous.

It should be noted that $\circ : G/H \times G/H \to G/H$ is continuous. In fact, let $A, B \in G/H$, $C = AB^{-1}$ and W' a neighborhood of C. Thus W' consists of elements zW, where W is a neighborhood in $(G; \circ)$ and $z \in W$. Because $C \in W'$, there exists an element $c \in W$ such that $C = cH$. Let $b \in B$ and $a = c \circ b$. Then $a \in A$. By definition, there are neighborhoods U, V of a and b respectively such that $UV^{-1} \subset W$ in $(G; \circ)$. Define

$$U' = \{\, xH \mid x \in U\} \quad \text{and} \quad V' = \{\, yH \mid y \in V\}.$$

There are neighborhoods of A and B in $(G/H; \circ)$, respectively. By $H \lhd G$, we get that

$$(xH)(yH)^{-1} = xHH^{-1}y = xHy^{-1} = (x \circ y^{-1})H \in W'.$$

Thus $U'(V')^{-1} \subset W'$, i.e., $\circ : G/H \times G/H \to G/H$ is continuous. Combining these discussions, we get the following result.

Theorem 9.4.2 *For any normal subgroup H of a topological group $(G; \circ)$, the quotient $(G/H; \circ)$ is a topological group.*

Such a topological group $(G/H; \circ)$ is called a *quotient topological group*.

9.4.4 Isomorphism Theorem. Let $(G; \circ)$, $(G'; \bullet)$ be topological groups and $f : (G; \circ) \to (G'; \bullet)$ be a mapping. If f is an algebraic homomorphism, also continuous, then f is called a *homomorphism* from topological group $(G; \circ)$ to $(G'; \bullet)$. Such a homomorphism is *opened* if it is an opened topological homeomorphism. Particularly, if f is an algebraic isomorphism and a homeomorphism, f is called an *isomorphism* from topological group $(G; \circ)$ to $(G'; \bullet)$.

Theorem 9.4.3 *Let $(G; \circ)$, $(G'; \bullet)$ be topological groups and $g : (G; \circ) \to (G'; \bullet)$ be an opened onto homomorphism,* Ker$g = N$. *Then N is a normal subgroup of $(G; \circ)$ and $(G; \circ)/N \simeq (G'; \bullet)$.*

Proof Clearly, N is closed by the continuous property of g and $N = \mathrm{Ker}g$. By Theorem 1.2.4, $\mathrm{Ker}g \lhd G$. Thus $(N;\circ)$ is a normal subgroup of $(G;\circ)$. Let $x' \in G'$. Then $g(x'N) = x'$. Define $f : G/N \to G'$ by $f(x'N) = x'$. We prove such a f is homeomorphism from topological space G to G/N.

Let $a' \in G'$, $f(a') = A$. Denoted by U' the a neighborhood of A in $(G/N;\circ)$. Then U' consists of cosets xN for $x \in U$, where U is a neighborhood of $(G;\circ)$. Let $a \in U$ such that $A = aN$. Since g is opened and $g(a) = a'$, there is a neighborhood V' of a' such that $g(U) \supset V'$. Now let $x' \in V'$. Then there is $x \in U$ such that $g(x) = x'$. Thus $f(x') = xN \in U'$, which implies that $f(V') \subset U'$, i.e., f is continuous.

Let $A = aN \in G/N$, $f^{-1}(A) = a'$ and U' a neighborhood of a'. Because g is continuous and $g(a) = a'$, there is a neighborhood of a such that $g(V) \subset U'$. Denoted by V' the neighborhood consisting of all cosets xN, where $x \in V$. Notice that $g(V) \subset U'$. We get that $f^{-1}(V') \subset U'$. Thus f^{-1} is also continuous.

Combining the above discussion, we know that $f : G/N \to G'$ is a homeomorphism. Notice that such a f is an isomorphism of algebraic group. We know it is an isomorphism of topological group by definition. □

9.4.5 Topological Multi-Group. A topological multi-group $(\mathscr{S}_G;\mathscr{O})$ is an algebraic multi-system $\left(\widetilde{\mathscr{A}};\mathscr{O}\right)$ with $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O} = \bigcup\limits_{i=1}^{m}\{\circ_i\}$ with conditions following hold:

(1) $(\mathscr{H}_i;\circ_i)$ is a group for each integer i, $1 \leq i \leq m$, namely, $(\mathscr{H},\mathscr{O})$ is a multi-group;

(2) $\widetilde{\mathscr{A}}$ is a combinatorially topological space $\mathscr{S}_G$, i.e., a combinatorial topological space underlying a structure G;

(3) the mapping $(a,b) \to a \circ b^{-1}$ is continuous for $\forall a,b \in \mathscr{H}_i$, $\forall \circ \in O_i$, $1 \leq i \leq m$.

For example, let $\mathbf{R}^{n_i}$, $1 \leq i \leq m$ be Euclidean spaces with an additive operation $+_i$ and scalar multiplication $\cdot$ determined by

$$(\lambda_1 \cdot x_1, \lambda_2 \cdot x_2, \cdots, \lambda_{n_i} \cdot x_{n_i}) +_i (\zeta_1 \cdot y_1, \zeta_2 \cdot y_2, \cdots, \zeta_{n_i} \cdot y_{n_i})$$
$$= (\lambda_1 \cdot x_1 + \zeta_1 \cdot y_1, \lambda_2 \cdot x_2 + \zeta_2 \cdot y_2, \cdots, \lambda_{n_i} \cdot x_{n_i} + \zeta_{n_i} \cdot y_{n_i})$$

for $\forall \lambda_l, \zeta_l \in \mathbf{R}$, where $1 \leq \lambda_l, \zeta_l \leq n_i$. Then each $\mathbf{R}^{n_i}$ is a continuous group under $+_i$. Whence, the algebraic multi-system $(\mathscr{E}_G(n_1,\cdots,n_m);\mathscr{O})$ is a topological multi-group with a underlying structure G by definition, where $\mathscr{O} = \bigcup\limits_{i=1}^{m}\{+_i\}$. Particularly, if $m = 1$, i.e., an n-dimensional Euclidean space $\mathbf{R}^n$ with the vector additive $+$ and multiplication $\cdot$ is a topological group.

A topological space S is *homogenous* if for $\forall a, b \in S$, there exists a continuous mapping $f : S \to S$ such that $f(b) = a$. We know a simple characteristic following.

Theorem 9.4.4 *If a topological multi-group $(\mathscr{S}_G; \mathscr{O})$ is arcwise connected and associative, then it is homogenous.*

Proof Notice that $\mathscr{S}_G$ is arcwise connected if and only if its underlying graph G is connected. For $\forall a, b \in \mathscr{S}_G$, without loss of generality, assume $a \in \mathscr{H}_0$ and $b \in \mathscr{H}_s$ and

$$P(a, b) = \mathscr{H}_0\mathscr{H}_1 \cdots \mathscr{H}_s, \quad s \geq 0,$$

a path from $\mathscr{H}_0$ to $\mathscr{H}_s$ in the graph G. Choose $c_1 \in \mathscr{H}_0 \cap \mathscr{H}_1$, $c_2 \in \mathscr{H}_1 \cap \mathscr{H}_2, \cdots$, $c_s \in \mathscr{H}_{s-1} \cap \mathscr{H}_s$. Then

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1}$$

is well-defined and

$$a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s b = a.$$

Let $L = a \circ_0 c_1 \circ_1 c_1^{-1} \circ_2 c_2 \circ_3 c_3 \circ_4 \cdots \circ_{s-1} c_s^{-1} \circ_s b^{-1} \circ_s$. Then L is continuous by the definition of topological multi-group. We finally get a continuous mapping $L : \mathscr{S}_G \to \mathscr{S}_G$ such that $L(b) = Lb = a$. Whence, $(\mathscr{S}_G; \mathscr{O})$ is homogenous. □

Corollary 9.4.1 *A topological group is homogenous if it is arcwise connected.*

A multi-subsystem $(\mathscr{L}_H; O)$ of $(\mathscr{S}_G; \mathscr{O})$ is called a *topological multi-subgroup* if it itself is a topological multi-group. Denoted by $\mathscr{L}_H \leq \mathscr{S}_G$. A criterion on topological multi-subgroups is shown in the following.

Theorem 9.4.5 *A multi-subsystem $(\mathscr{L}_H; O_1)$ is a topological multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$, where $O_1 \subset O$ if and only if it is a multi-subgroup of $(\mathscr{S}_G; \mathscr{O})$ in algebra.*

Proof The necessity is obvious. For the sufficiency, we only need to prove that for any operation $\circ \in O_1$, $a \circ b^{-1}$ is continuous in $\mathscr{L}_H$. Notice that the condition (3) in the definition of topological multi-group can be replaced by:

for any neighborhood $N_{\mathscr{S}_G}(a \circ b^{-1})$ of $a \circ b^{-1}$ in $\mathscr{S}_G$, there always exist neighborhoods $N_{\mathscr{S}_G}(a)$ and $N_{\mathscr{S}_G}(b^{-1})$ of a and b^{-1} such that $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) \subset N_{\mathscr{S}_G}(a \circ b^{-1})$, where $N_{\mathscr{S}_G}(a) \circ N_{\mathscr{S}_G}(b^{-1}) = \{x \circ y | \forall x \in N_{\mathscr{S}_G}(a), y \in N_{\mathscr{S}_G}(b^{-1})\}$

by the definition of mapping continuity. Whence, we only need to show that for any neighborhood $N_{\mathscr{L}_H}(x\circ y^{-1})$ in $\mathscr{L}_H$, where $x, y \in \mathscr{L}_H$ and $\circ \in O_1$, there exist neighborhoods $N_{\mathscr{L}_H}(x)$ and $N_{\mathscr{L}_H}(y^{-1})$ such that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x \circ y^{-1})$ in $\mathscr{L}_H$. In fact, each neighborhood $N_{\mathscr{L}_H}(x\circ y^{-1})$ of $x\circ y^{-1}$ can be represented by a form $N_{\mathscr{S}_G}(x\circ y^{-1})\cap \mathscr{L}_H$. By assumption, $(\mathscr{S}_G; \mathscr{O})$ is a topological multi-group, we know that there are neighborhoods $N_{\mathscr{S}_G}(x)$, $N_{\mathscr{S}_G}(y^{-1})$ of x and y^{-1} in $\mathscr{S}_G$ such that $N_{\mathscr{S}_G}(x) \circ N_{\mathscr{S}_G}(y^{-1}) \subset N_{\mathscr{S}_G}(x \circ y^{-1})$. Notice that $N_{\mathscr{S}_G}(x)\cap\mathscr{L}_H$, $N_{\mathscr{S}_G}(y^{-1})\cap\mathscr{L}_H$ are neighborhoods of x and y^{-1} in $\mathscr{L}_H$. Now let $N_{\mathscr{L}_H}(x) = N_{\mathscr{S}_G}(x) \cap \mathscr{L}_H$ and $N_{\mathscr{L}_H}(y^{-1}) = N_{\mathscr{S}_G}(y^{-1}) \cap \mathscr{L}_H$. Then we get that $N_{\mathscr{L}_H}(x) \circ N_{\mathscr{L}_H}(y^{-1}) \subset N_{\mathscr{L}_H}(x\circ y^{-1})$ in $\mathscr{L}_H$, i.e., the mapping $(x, y) \to x\circ y^{-1}$ is continuous. Whence, $(\mathscr{L}_H; O_1)$ is a topological multi-subgroup. □

Particularly, for the topological groups, we know the following consequence.

Corollary 9.4.2 *A subset of a topological group* $(\Gamma; \circ)$ *is a topological subgroup if and only if it is a closed subgroup of* $(\Gamma; \circ)$ *in algebra.*

For two topological multi-groups $(\mathscr{S}_{G_1}; \mathscr{O}_1)$ and $(\mathscr{S}_{G_2}; \mathscr{O}_2)$, a mapping $\omega : (\mathscr{S}_{G_1}; \mathscr{O}_1) \to (\mathscr{S}_{G_2}; \mathscr{O}_2)$ is a *homomorphism* if it satisfies the following conditions:

(1) ω is a homomorphism from multi-groups $(\mathscr{S}_{G_1}; \mathscr{O}_1)$ to $(\mathscr{S}_{G_2}; \mathscr{O}_2)$, namely, for $\forall a, b \in \mathscr{S}_{G_1}$ and $\circ \in O_1$, $\omega(a \circ b) = \omega(a)\omega(\circ)\omega(b)$;

(2) ω is a continuous mapping from topological spaces $\mathscr{S}_{G_1}$ to $\mathscr{S}_{G_1}$, i.e., for $\forall x \in \mathscr{S}_{G_1}$ and a neighborhood U of $\omega(x)$, $\omega^{-1}(U)$ is a neighborhood of x.

Furthermore, if $\omega : (\mathscr{S}_{G_1}; \mathscr{O}_1) \to (\mathscr{S}_{G_2}; \mathscr{O}_2)$ is an isomorphism in algebra and a homeomorphism in topology, then it is called an *isomorphism*, particularly, an *automorphism* if $(\mathscr{S}_{G_1}; \mathscr{O}_1) = (\mathscr{S}_{G_2}; \mathscr{O}_2)$ between topological multi-groups $(\mathscr{S}_{G_1}; \mathscr{O}_1)$ and $(\mathscr{S}_{G_2}; \mathscr{O}_2)$.

§9.5 COMBINATORIAL METRIC SPACES

9.5.1 Multi-Metric Space. A multi-metric space is a union $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ such that each M_i is a space with a metric ρ_i for $\forall i, 1 \leq i \leq m$. Usually, as we say a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, it means that a multi-metric space with metrics $\rho_1, \rho_2, \cdots, \rho_m$ such that (M_i, ρ_i) is a metric space for any integer $i, 1 \leq i \leq m$. For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, $x \in \widetilde{M}$

and a positive number r, a *r-disk* $B(x,r)$ in $\widetilde{M}$ is defined by

$$B(x,r) = \{\, y \mid \text{there exists an integer } k, 1 \le k \le m \text{ such that } \rho_k(y,x) < r, y \in \widetilde{M}\}$$

Remark 9.5.1 The following two extremal cases are permitted in multi-metric spaces:

(1) there are integers $i_1, i_2, \cdots, i_s$ such that $M_{i_1} = M_{i_2} = \cdots = M_{i_s}$, where $i_j \in \{1,2,\cdots,m\}$, $1 \le j \le s$;

(2) there are integers $l_1, l_2, \cdots, l_s$ such that $\rho_{l_1} = \rho_{l_2} = \cdots = \rho_{l_s}$, where $l_j \in \{1,2,\cdots,m\}$, $1 \le j \le s$.

Theorem 9.5.1 *Let $\rho_1, \rho_2, \cdots, \rho_m$ be m metrics on a space M and let F be a function on $\mathbf{R}^m$ such that the following conditions hold:*

(1) $F(x_1, x_2, \cdots, x_m) \ge F(y_1, y_2, \cdots, y_m)$ *for* $\forall i, 1 \le i \le m, x_i \ge y_i$*;*

(2) $F(x_1, x_2, \cdots, x_m) = 0$ *only if* $x_1 = x_2 = \cdots = x_m = 0$*;*

(3) *for two m-tuples* $(x_1, x_2, \cdots, x_m)$ *and* $(y_1, y_2, \cdots, y_m)$*,*

$$F(x_1, x_2, \cdots, x_m) + F(y_1, y_2, \cdots, y_m) \ge F(x_1 + y_1, x_2 + y_2, \cdots, x_m + y_m).$$

Then $F(\rho_1, \rho_2, \cdots, \rho_m)$ is also a metric on M.

Proof We only need to prove that $F(\rho_1, \rho_2, \cdots, \rho_m)$ satisfies those conditions of metric for $\forall x, y, z \in M$. By the condition (2), $F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = 0$ only if $\rho_i(x,y) = 0$ for any integer i. Since ρ_i is a metric on M, we know that $x = y$.

For any integer i, $1 \le i \le m$, since ρ_i is a metric on M, we know that $\rho_i(x,y) = \rho_i(y,x)$. Whence,

$$F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = F(\rho_1(y,x), \rho_2(y,x), \cdots, \rho_m(y,x)).$$

Now by (1) and (3), we get that

$$\begin{aligned} & F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) + F(\rho_1(y,z), \rho_2(y,z), \cdots, \rho_m(y,z)) \\ & \ge F(\rho_1(x,y) + \rho_1(y,z), \rho_2(x,y) + \rho_2(y,z), \cdots, \rho_m(x,y) + \rho_m(y,z)) \\ & \ge F(\rho_1(x,z), \rho_2(x,z), \cdots, \rho_m(x,z)). \end{aligned}$$

Therefore, $F(\rho_1, \rho_2, \cdots, \rho_m)$ is a metric on M. □

Corollary 9.5.1 If $\rho_1, \rho_2, \cdots, \rho_m$ are m metrics on a space M, then $\rho_1 + \rho_2 + \cdots + \rho_m$ and $\dfrac{\rho_1}{1+\rho_1} + \dfrac{\rho_2}{1+\rho_2} + \cdots + \dfrac{\rho_m}{1+\rho_m}$ are also metrics on M.

9.5.2 Convergent Sequence in Multi-Metric Space. A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is said to be *convergent to a point* $x, x \in \widetilde{M}$ if for any number $\epsilon > 0$, there exist numbers N and $i, 1 \leq i \leq m$ such that

$$\rho_i(x_n, x) \; < \; \epsilon$$

provided $n \geq N$. If $\{x_n\}$ is convergent to a point $x, x \in \widetilde{M}$, we denote it by $\lim\limits_{n} x_n = x$.

We get a characteristic for convergent sequences in multi-metric spaces following.

Theorem 9.5.2 *A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if there exist integers N and $k, 1 \leq k \leq m$ such that the subsequence $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k).*

Proof If there exist integers N and $k, 1 \leq k \leq m$ such that $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k), then for any number $\epsilon > 0$, by definition there exist an integer P and a point $x, x \in M_k$ such that

$$\rho_k(x_n, x) \; < \; \epsilon$$

if $n \geq max\{N, \; P\}$.

Now if $\{x_n\}$ is a convergent sequence in the multi-space $\widetilde{M}$, by definition for any positive number $\epsilon > 0$, there exist a point $x, x \in \widetilde{M}$, natural numbers $N(\epsilon)$ and integer $k, 1 \leq k \leq m$ such that if $n \geq N(\epsilon)$, then

$$\rho_k(x_n, x) \; < \; \epsilon.$$

Thus $\{x_n | n \geq N(\epsilon)\} \subset M_k$ and $\{x_n | n \geq N(\epsilon)\}$ is a convergent sequence in (M_k, ρ_k). □

Theorem 9.5.3 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a multi-metric space. For two sequences $\{x_n\}$, $\{y_n\}$ in $\widetilde{M}$, if $\lim\limits_{n} x_n = x_0$, $\lim\limits_{n} y_n = y_0$ and there is an integer p such that $x_0, y_0 \in M_p$, then $\lim\limits_{n} \rho_p(x_n, y_n) = \rho_p(x_0, y_0)$.*

Proof According to Theorem 9.5.2, there exist integers N_1 and N_2 such that if $n \geq max\{N_1, N_2\}$, then $x_n, y_n \in M_p$. Whence,

$$\rho_p(x_n, y_n) \leq \rho_p(x_n, x_0) + \rho_p(x_0, y_0) + \rho_p(y_n, y_0)$$

and

$$\rho_p(x_0, y_0) \leq \rho_p(x_n, x_0) + \rho_p(x_n, y_n) + \rho_p(y_n, y_0).$$

Therefore,

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ \leq \ \rho_p(x_n, x_0) + \rho_p(y_n, y_0).$$

Now for any number $\epsilon > 0$, since $\lim\limits_{n} x_n = x_0$ and $\lim\limits_{n} y_n = y_0$, there exist numbers $N_1(\epsilon), N_1(\epsilon) \geq N_1$ and $N_2(\epsilon), N_2(\epsilon) \geq N_2$ such that $\rho_p(x_n, x_0) \leq \dfrac{\epsilon}{2}$ if $n \geq N_1(\epsilon)$ and $\rho_p(y_n, y_0) \leq \dfrac{\epsilon}{2}$ if $n \geq N_2(\epsilon)$. Whence, if we choose $n \geq max\{N_1(\epsilon), N_2(\epsilon)\}$, then

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ < \ \epsilon.$$ □

Can a convergent sequence has more than one limiting points? The following result answers this question.

Theorem 9.5.4 *If $\{x_n\}$ is a convergent sequence in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, then $\{x_n\}$ has only one limit point.*

Proof According to Theorem 9.5.2, there exist integers N and $i, 1 \leq i \leq m$ such that $x_n \in M_i$ if $n \geq N$. Now if

$$\lim_{n} x_n = x_1 \text{ and } \lim_{n} x_n = x_2,$$

and $n \geq N$, by definition,

$$0 \leq \rho_i(x_1, x_2) \leq \rho_i(x_n, x_1) + \rho_i(x_n, x_2).$$

Thus $\rho_i(x_1, x_2) = 0$. Consequently, $x_1 = x_2$. □

Theorem 9.5.5 *Any convergent sequence in a multi-metric space is a bounded points set.*

Proof According to Theorem 9.5.4, we obtain this result immediately. □

9.5.3 Completed Sequence in Multi-Metric Space. A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is called a *Cauchy sequence* if for any number $\epsilon > 0$, there exist integers $N(\epsilon)$ and $s, 1 \leq s \leq m$ such that for any integers $m, n \geq N(\epsilon)$, $\rho_s(x_m, x_n) < \epsilon$.

Theorem 9.5.6 *A Cauchy sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if $|\{x_n\} \bigcap M_k|$ is finite or infinite but $\{x_n\} \bigcap M_k$ is convergent in (M_k, ρ_k) for $\forall k, 1 \leq k \leq m$.*

Proof The necessity of conditions in this theorem is known by Theorem 9.5.2. Now we prove the sufficiency. By definition, there exist integers $s, 1 \leq s \leq m$ and N_1 such that $x_n \in M_s$ if $n \geq N_1$. Whence, if $|\{x_n\} \bigcap M_k|$ is infinite and $\lim\limits_{n} \{x_n\} \bigcap M_k = x$, then there must be $k = s$. Denote by $\{x_n\} \bigcap M_k = \{x_{k1}, x_{k2}, \cdots, x_{kn}, \cdots\}$.

For any positive number $\epsilon > 0$, there exists an integer $N_2, N_2 \geq N_1$ such that $\rho_k(x_m, x_n) < \dfrac{\epsilon}{2}$ and $\rho_k(x_{kn}, x) < \dfrac{\epsilon}{2}$ if $m, n \geq N_2$. According to Theorem 9.5.3, we get that

$$\rho_k(x_n, x) \leq \rho_k(x_n, x_{kn}) + \rho_k(x_{kn}, x) < \epsilon$$

if $n \geq N_2$. Whence, $\lim\limits_{n} x_n = x$. □

A multi-metric space $\widetilde{M}$ is said to be *completed* if its every Cauchy sequence is convergent. For a completed multi-metric space, we obtain two important results similar to Theorems 1.5.3 and 1.5.4 in metric spaces.

Theorem 9.5.7 *Let* $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ *be a completed multi-metric space. For an* ϵ*-disk sequence* $\{B(\epsilon_n, x_n)\}$, *where* $\epsilon_n > 0$ *for* $n = 1, 2, 3, \cdots$, *if the following conditions hold:*

(1) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(2) $\lim\limits_{n\to+\infty} \epsilon_n = 0$,

then $\bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$ *only has one point.*

Proof First, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. By the condition (1), we know that if $m \geq n$, then $x_m \in B(\epsilon_m, x_m) \subset B(\epsilon_n, x_n)$. Whence $\rho_i(x_m, x_n) < \epsilon_n$ provided $x_m, x_n \in M_i$ for $\forall i, 1 \leq i \leq m$.

Now for any positive number ϵ, since $\lim\limits_{n\to+\infty} \epsilon_n = 0$, there exists an integer $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $\epsilon_n < \epsilon$. Therefore, if $x_n \in M_l$, then $\lim\limits_{m\to+\infty} x_m = x_n$. Thereby there exists an integer N such that if $m \geq N$, then $x_m \in M_l$ by Theorem 9.5.2. Choice integers $m, n \geq max\{N, N(\epsilon)\}$, we know that

$$\rho_l(x_m, x_n) < \epsilon_n < \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence.

By the assumption that $\widetilde{M}$ is completed, we know that the sequence $\{x_n\}$ is convergent to a point x_0, $x_0 \in \widetilde{M}$. By conditions of (i) and (ii), we get that $\rho_l(x_0, x_n) < \epsilon_n$ if $m \to +\infty$. Whence, $x_0 \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$.

Now if there is a point $y \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$, then there must be $y \in M_l$. We get that

$$0 \leq \rho_l(y, x_0) = \lim_{n} \rho_l(y, x_n) \leq \lim_{n\to+\infty} \epsilon_n = 0$$

by Theorem 9.5.3. Thus $\rho_l(y, x_0) = 0$. By the definition of metric function, we get that $y = x_0$. □

Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be two multi-metric spaces and let $f : \widetilde{M}_1 \rightarrow \widetilde{M}_2$ be a mapping, $x_0 \in \widetilde{M}_1$, $f(x_0) = y_0$. For $\forall\epsilon > 0$, if there exists a number δ such that $f(x) = y \in B(\epsilon, y_0) \subset \widetilde{M}_2$ for $\forall x \in B(\delta, x_0)$, i.e.,

$$f(B(\delta, x_0)) \subset B(\epsilon, y_0),$$

then f is called *continuous at point* x_0. A mapping $f : \widetilde{M}_1 \rightarrow \widetilde{M}_2$ is called a *continuous mapping* from $\widetilde{M}_1$ to $\widetilde{M}_2$ if f is continuous at every point of $\widetilde{M}_1$.

For a continuous mapping f from $\widetilde{M}_1$ to $\widetilde{M}_2$ and a convergent sequence $\{x_n\}$ in $\widetilde{M}_1$, $\lim\limits_n x_n = x_0$, we can prove that

$$\lim_n f(x_n) = f(x_0).$$

For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ and a mapping $T : \widetilde{M} \rightarrow \widetilde{M}$, if there is a point $x^* \in \widetilde{M}$ such that $Tx^* = x^*$, then x^* is called a *fixed point* of T. Denote the number of fixed points of a mapping T in $\widetilde{M}$ by ${}^{\#}\Phi(T)$. A mapping T is called a *contraction* on a multi-metric space $\widetilde{M}$ if there are a constant $\alpha, 0 < \alpha < 1$ and integers $i, j, 1 \leq i, j \leq m$ such that for $\forall x, y \in M_i$, $Tx, Ty \in M_j$ and

$$\rho_j(Tx, Ty) \leq \alpha\rho_i(x, y).$$

Theorem 9.5.8 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space and let T be a contraction on $\widetilde{M}$. Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Proof Choose arbitrary points $x_0, y_0 \in M_1$ and define recursively

$$x_{n+1} = Tx_n, \quad y_{n+1} = Tx_n$$

for $n = 1, 2, 3, \cdots$. By definition, we know that for any integer $n, n \geq 1$, there exists an integer $i, 1 \leq i \leq m$ such that $x_n, y_n \in M_i$. Whence, we inductively get that

$$0 \leq \rho_i(x_n, y_n) \leq \alpha^n\rho_1(x_0, y_0).$$

Notice that $0 < \alpha < 1$, we know that $\lim\limits_{n\rightarrow+\infty} \alpha^n = 0$. Thereby there exists an integer i_0 such that

$$\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0.$$

Therefore, there exists an integer N_1 such that $x_n, y_n \in M_{i_0}$ if $n \geq N_1$. Now if $n \geq N_1$, we get that

$$\begin{aligned}\rho_{i_0}(x_{n+1}, x_n) &= \rho_{i_0}(Tx_n, Tx_{n-1})\\ &\leq \alpha\rho_{i_0}(x_n, x_{n-1}) = \alpha\rho_{i_0}(Tx_{n-1}, Tx_{n-2})\\ &\leq \alpha^2\rho_{i_0}(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^{n-N_1}\rho_{i_0}(x_{N_1+1}, x_{N_1}).\end{aligned}$$

and generally,

$$\begin{aligned}\rho_{i_0}(x_m, x_n) &\leq \rho_{i_0}(x_n, x_{n+1}) + \rho_{i_0}(x_{n+1}, x_{n+2}) + \cdots + \rho_{i_0}(x_{n-1}, x_n)\\ &\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\rho_{i_0}(x_{N_1+1}, x_{N_1})\\ &\leq \frac{\alpha^n}{1-\alpha}\rho_{i_0}(x_{N_1+1}, x_{N_1}) \to 0 \quad (m, n \to +\infty).\end{aligned}$$

for $m \geq n \geq N_1$. Therefore, $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. Similarly, we can also prove $\{y_n\}$ is a Cauchy sequence.

Because $\widetilde{M}$ is a completed multi-metric space, we know that

$$\lim_n x_n = \lim_n y_n = z^*.$$

Now we prove z^* is a fixed point of T in $\widetilde{M}$. In fact, by $\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0$, there exists an integer N such that

$$x_n, y_n, Tx_n, Ty_n \in M_{i_0}$$

if $n \geq N + 1$. Whence,

$$\begin{aligned}0 \leq \rho_{i_0}(z^*, Tz^*) &\leq \rho_{i_0}(z^*, x_n) + \rho_{i_0}(y_n, Tz^*) + \rho_{i_0}(x_n, y_n)\\ &\leq \rho_{i_0}(z^*, x_n) + \alpha\rho_{i_0}(y_{n-1}, z^*) + \rho_{i_0}(x_n, y_n).\end{aligned}$$

Notice that

$$\lim_{n\to+\infty}\rho_{i_0}(z^*, x_n) = \lim_{n\to+\infty}\rho_{i_0}(y_{n-1}, z^*) = \lim_{n\to+\infty}\rho_{i_0}(x_n, y_n) = 0.$$

We get $\rho_{i_0}(z^*, Tz^*) = 0$, i.e., $Tz^* = z^*$.

For other chosen points $u_0, v_0 \in M_1$, we can also define recursively $u_{n+1} = Tu_n,\ v_{n+1} = Tv_n$ and get a limiting point $\lim_n u_n = \lim_n v_n = u^* \in M_{i_0}, Tu^* \in M_{i_0}$. Since

$$\rho_{i_0}(z^*, u^*) = \rho_{i_0}(Tz^*, Tu^*) \leq \alpha\rho_{i_0}(z^*, u^*)$$

and $0 < \alpha < 1$, there must be $z^* = u^*$.

Similarly, consider the points in $M_i, 2 \leq i \leq m$. We get that

$$1 \leq^{\#} \Phi(T) \leq m. \qquad \square$$

Particularly, let $m = 1$. We get *Banach theorem* in metric spaces following.

Corollary 9.5.2(Banach) *Let M be a metric space and let T be a contraction on M. Then T has just one fixed point.*

§9.6 **RESEARCH PROBLEMS**

9.6.1 The *mathematical combinatorics*, particularly, *spacial combinatorics* is a universal theory for advancing mathematical sciences on CC conjecture [Mao19], a philosophical thought on mathematics. Applications of this thought can be found in references [Mao10]-[Mao11], [Mao17]-[Mao38], Particularly, these monographs [Mao37]-[Mao38].

9.6.2 The inherited graph of a multi-space $\widetilde{S} = \bigcup\limits_{i=1}^{n} \Sigma_i$ is uniquely determined by Definition 9.1.1, which enables one to classify multi-space combinatorially. The central problem is to find an applicable labeling l^E, i.e., the characteristic ϖ on $\Sigma_i \bigcap \Sigma_j$.

Problem 9.6.1 *Characterize multi-spaces with an inherited graph $G \in \mathscr{L}$, where $\mathscr{L}$ is a family of graphs, such as those of trees, Euler graphs, Hamiltonian graphs, factorable graphs, n-colorable graphs, $\cdots$, etc..*

Problem 9.6.2 *Characterize inherited graphs of multi-systems $\left(\widetilde{A}; \widetilde{O}\right)$ with $\widetilde{A} = \bigcup\limits_{i=1}^{n} A_i$ and $\widetilde{O} = \{\circ_i | 1 \leq i \leq n\}$ such that $(A_i; \circ_i)$ is a well-known algebraic system for integers $1 \leq i \leq n$, for instance, simple group, Sylow group, cyclic group, $\cdots$, etc..*

9.6.3 Similarly, consider Problems 9.6.1-9.6.2 for combinatorial Euclidean spaces.

Problem 9.6.3 *Characterize a combinatorial Euclidean space underlying graph G and calculate it characteristic, for example, dimension, isometry, $\cdots$, etc..*

9.6.4 For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorial C^h-differential manifold $\left(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}}\right)$ is a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\widetilde{M}(n_1, n_2, \cdots, n_m) = \bigcup\limits_{i \in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

(2) For $\forall\alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the overlap maps

$$\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta\left(U_\alpha \bigcap U_\beta\right) \to \varphi_\beta(U_\beta) \ \text{ and } \ \varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha\left(U_\alpha \bigcap U_\beta\right) \to \varphi_\alpha(U_\alpha)$$

are C^h-mappings, such as those shown in Fig.9.6.1 following.

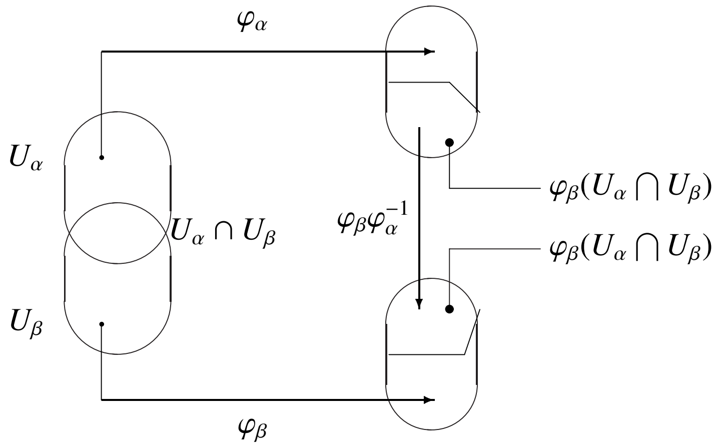

φ_α
U_α
$\varphi_\beta(U_\alpha \bigcap U_\beta)$
$U_\alpha \cap U_\beta$
$\varphi_\beta\varphi_\alpha^{-1}$
$\varphi_\beta(U_\alpha \bigcap U_\beta)$
U_β
φ_β

Fig.9.6.1

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U; \varphi)$ is a local chart of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U; \varphi) \in \widetilde{\mathcal{A}}$.

Such a combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is said to be *smooth* if it is endowed with a C^∞-differential structure. Let $\widetilde{\mathcal{A}}$ be an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. Choose a local chart $(U; \varpi)$ in $\widetilde{\mathcal{A}}$. For $\forall p \in (U; \varphi)$, if $\varpi_p : U_p \to \bigcup\limits_{i=1}^{s(p)} B^{n_i(p)}$ and $\widehat{s}(p) = \dim(\bigcap\limits_{i=1}^{s(p)} B^{n_i(p)})$, the following $s(p) \times n_{s(p)}$ matrix $[\varpi(p)]$

$$[\varpi(p)] = \left[\begin{array}{cccccccc} \frac{x^{11}}{s(p)} & \cdots & \frac{x^{1\widehat{s}(p)}}{s(p)} & x^{1(\widehat{s}(p)+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ \frac{x^{21}}{s(p)} & \cdots & \frac{x^{2\widehat{s}(p)}}{s(p)} & x^{2(\widehat{s}(p)+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{x^{s(p)1}}{s(p)} & \cdots & \frac{x^{s(p)\widehat{s}(p)}}{s(p)} & x^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & x^{s(p)n_{s(p)}-1} & x^{s(p)n_{s(p)}} \end{array}\right]$$

with $x^{is} = x^{js}$ for $1 \leq i, j \leq s(p), 1 \leq s \leq \widehat{s}(p)$ is called the *coordinate matrix of p*. For emphasize ϖ is a matrix, we often denote local charts in a combinatorial differential manifold by $(U; [\varpi])$. Applying the coordinate matrix system of a combinatorial differential manifold $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$, we can define C^h mappings, functions and establish differential theory on combinatorial manifolds. The reader is refereed to [Mao33] or [Mao38] for details.

9.6.5 Besides topological multi-groups, there are also topological multi-rings and multi-fields in mathematics. A distributive multi-system $\left(\widetilde{\mathscr{A}}; \mathscr{O}_1 \hookrightarrow \mathscr{O}_2\right)$ with $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$, $\mathscr{O}_1 = \bigcup\limits_{i=1}^{m}\{\cdot_i\}$ and $\mathscr{O}_2 = \bigcup\limits_{i=1}^{m}\{+_i\}$ is called a topological multi-ring if

(1) $(\mathscr{H}_i; +_i, \cdot_i)$ is a ring for each integer i, $1 \leq i \leq m$, i.e., $(\mathscr{H}, \mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is a multi-ring;

(2) $\widetilde{\mathscr{A}}$ is a combinatorially topological space $\mathscr{S}_G$;

(3) the mappings $(a, b) \rightarrow a \cdot_i b^{-1}$, $(a, b) \rightarrow a +_i (-_i b)$ are continuous for $\forall a, b \in \mathscr{H}_i$, $1 \leq i \leq m$.

Denoted by $(\mathscr{S}_G; \mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ a topological multi-ring. A topological multi-ring $(\mathscr{S}_G; \mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ is called a *topological divisible multi-ring* or *multi-field* if the condition (1) is replaced by $(\mathscr{H}_i; +_i, \cdot_i)$ *is a divisible ring or field for each integer* $1 \leq i \leq m$. Particularly, if $m = 1$, then a topological multi-ring, divisible multi-ring or multi-field is nothing but a topological ring, divisible ring or field in mathematics, i.e., a ring, divisible ring or field $(R; +, \cdot)$ such that

(1) R is a topological space;

(2) the mappings $(a, b) \rightarrow a \cdot b^{-1}$, $(a, b) \rightarrow a - b)$ are continuous for $\forall a, b \in R$.

More results for topological groups, topological rings can be found in [Pon1] or [Pon2]. The reader is refereed to [Mao30], [Mao33] or [Mao38] for results on topological multi-groups, topological multi-rings and topological multi-fields.

9.6.6 Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space underlying graph G and T a contraction on $\widetilde{M}$. We have know that $1 \leq {}^{\#}\Phi(T) \leq m$ by Theorem 9.5.8. Such result is holds for any multi-metric space. Generally, there is an open problem on the number of fixed points of a contraction on multi-metric spaces following.

Problem 9.6.4 *Generalize Banach's fixed point theorem, or determine the lower and upper boundary of* ${}^{\#}\Phi(T)$ *for contractions* T *on a completed multi-metric space underlying a graph G, such as those of tree, circuit, completed graph,* 1*-factorable graph,* $\cdots$*, etc..*

CHAPTER 10.

Applications

There are many simpler but more puzzling questions confused the eyes of human beings thousands years and does not know an answer even until today. For example, *Whether are there finite, or infinite cosmoses? Is there just one? What is the dimension of our cosmos?* The dimension of cosmos in eyes of the ancient Greeks is 3, but Einstein's is 4. In recent decades, 10 or 11 is the dimension of cosmos in superstring theory or M-theory. All these assumptions acknowledge that there is just one cosmos. *Which one is the correct?* We have known that the Smarandache multi-space is a systematic notion dealing with objective, particularly for one knowing the WORLD. Thus it is the best candidate for the *Theory of Everything*, i.e., a fundamental united theory of all physical phenomena in nature. For introducing the effect of Smarandache multi-space to sciences, the applications of Smarandache multi-spaces to physics, particularly, the relativity theory with Schwarzschild spacetime, to generalizing the input-output model for economy analysis and to knowing well infection rule for decreasing or eliminating infectious disease are presented in this chapter.

§10.1 PSEUDO-FACES OF SPACES

10.1.1 Pseudo-Face. For find different representations of a Euclidean space $\mathbf{R}^n$, we introduce the conception of pseudo-face following.

Definition 10.1.1 *Let* $\mathbf{R}^m$ *be a Euclid space and* $(\mathbf{R}^n, \omega)$ *a Euclidean pseudo-space. If there is a continuous mapping* $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$*, then the pseudo-metric space* $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ *is called a pseudo-face of* $\mathbf{R}^m$ *in* $(\mathbf{R}^n, \omega)$*.*

For example, these pseudo-faces of $\mathbf{R}^3$ in $\mathbf{R}^2$ have been discussed in Chapter 8. For the existence of pseudo-faces of a Euclid space $\mathbf{R}^m$ in $\mathbf{R}^n$, we know a result following.

Theorem 10.1.1 *Let* $\mathbf{R}^m$ *be a Euclid space and* $(\mathbf{R}^n, \omega)$ *a Euclidean pseudo-space. Then there exists a pseudo-face of* $\mathbf{R}^m$ *in* $(\mathbf{R}^n, \omega)$ *if and only if for any number* $\epsilon > 0$*, there exists a number* $\delta > 0$ *such that for* $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ *with* $\|\overline{u} - \overline{v}\| < \delta$*,*

$$\|\omega(p(\overline{u})) - \omega(p(\overline{v}))\| < \epsilon,$$

where $\|\overline{u}\|$ *denotes the norm of vector* $\overline{u}$ *in Euclid spaces.*

Proof We show that there exists a continuous mapping $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ if and only if all of these conditions hold. By the definition of Euclidean pseudo-space $(\mathbf{R}^n, \omega)$, ω is continuous. We know that for any number $\epsilon > 0$, $\|\omega(\overline{x}) - \omega(\overline{y})\| < \epsilon$ for $\forall \overline{x}, \overline{y} \in \mathbf{R}^n$ if and only if there exists a number $\delta_1 > 0$ such that $\|\overline{x} - \overline{y})\| < \delta_1$.

By definition, a mapping $q : \mathbf{R}^m \to \mathbf{R}^n$ is continuous if and only if for any number $\delta_1 > 0$, there exists a number $\delta_2 > 0$ such that $\|q(\overline{x}) - q(\overline{y})\| < \delta_1$ for $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ with $\|\overline{u} - \overline{v})\| < \delta_2$. Whence, $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ is continuous if and only if for any number $\epsilon > 0$, there is a number $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\|\omega(p(\overline{u})) - \omega(p(\overline{v}))\| < \epsilon$$

for $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ with $\|\overline{u} - \overline{v})\| < \delta$. □

Corollary 10.1.1 *If* $m \geq n + 1$*, let* $\omega : \mathbf{R}^n \to \mathbf{R}^{m-n}$ *be a continuous mapping, then* $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ *is a pseudo-face of* $\mathbf{R}^m$ *in* $(\mathbf{R}^n, \omega)$ *with*

$$p(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_m) = \omega(x_1, x_2, \cdots, x_n).$$

Particularly, if $m = 3, n = 2$ *and* ω *is an angle function, then* $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ *is a pseudo-face with* $p(x_1, x_2, x_3) = \omega(x_1, x_2)$*.*

A relation for a continuous mapping of Euclid space and that of between pseudo-faces is established in the next.

Theorem 10.1.2 *Let* $g : \mathbf{R}^m \to \mathbf{R}^m$ *and* $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ *be continuous mappings. Then* $pgp^{-1} : (\mathbf{R}^n, \omega) \to (\mathbf{R}^n, \omega)$ *is also a continuous mapping.*

Proof Because the composition of continuous mappings is also a continuous mapping, we know that pgp^{-1} is continuous.

Now for $\forall \omega(x_1, x_2, \cdots, x_n) \in (\mathbf{R}^n, \omega)$, assume that $p(y_1, y_2, \cdots, y_m) = \omega(x_1, x_2, \cdots, x_n)$, $g(y_1, y_2, \cdots, y_m) = (z_1, z_2, \cdots, z_m)$ and $p(z_1, z_2, \cdots, z_m) = \omega(t_1, t_2, \cdots, t_n)$. Calculation shows that

$$\begin{aligned} pgp^{-}(\omega(x_1, x_2, \cdots, x_n)) &= pg(y_1, y_2, \cdots, y_m) \\ &= p(z_1, z_2, \cdots, z_m) = \omega(t_1, t_2, \cdots, t_n) \in (\mathbf{R}^n, \omega). \end{aligned}$$

Whence, $pgp^{-1} : (\mathbf{R}^n, \omega) \to (\mathbf{R}^n, \omega)$ is continuous. □

Corollary 10.1.2 *Let* $C(\mathbf{R}^m)$ *and* $C(\mathbf{R}^n, \omega)$ *be sets of continuous mapping on Euclid space* $\mathbf{R}^m$ *and pseudo-metric space* $(\mathbf{R}^n, \omega)$*, respectively. If there is a Euclidean pseudo-space for* $\mathbf{R}^m$ *in* $(\mathbf{R}^n, \omega)$*. Then there is a bijection between* $C(\mathbf{R}^m)$ *and* $C(\mathbf{R}^n, \omega)$*.*

10.1.2 Pseudo-Shape. For an object $\mathcal{B}$ in a Euclid space $\mathbf{R}^m$, its shape in a pseudo-face $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ of $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$ is called a *pseudo-shape* of $\mathcal{B}$. We get results for pseudo-shapes of balls in the following.

Theorem 10.1.3 *Let* $\mathcal{B}$ *be an* $(n+1)$*-ball of radius* R *in a space* $\mathbf{R}^{n+1}$*, i.e.,*

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \le R^2.$$

Define a continuous mapping $\omega : \mathbf{R}^n \to \mathbf{R}^n$ *by*

$$\omega(x_1, x_2, \cdots, x_n) = \varsigma t(x_1, x_2, \cdots, x_n)$$

for a real number ς *and a continuous mapping* $p : \mathbf{R}^{n+1} \to \mathbf{R}^n$ *by*

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n).$$

Then the pseudo-shape of $\mathcal{B}$ *in* $(\mathbf{R}^n, \omega)$ *is a ball of radius* $\dfrac{\sqrt{R^2 - t^2}}{\varsigma t}$ *for any parameter* $t, -R \le t \le R$*. Particularly, if* $n = 2$ *and* $\varsigma = \dfrac{1}{2}$*, it is a circle of radius* $\sqrt{R^2 - t^2}$ *for parameter* t *and an elliptic ball in* $\mathbf{R}^3$ *as shown in Fig.*10.1.1.

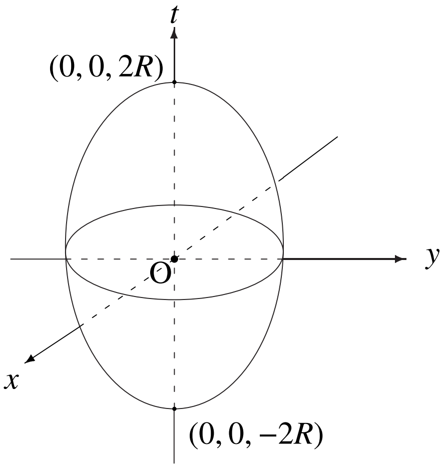


Fig.10.1.1

Proof For any parameter t, an $(n+1)$-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \leq R^2$$

can be transferred to an n-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2 - t^2$$

of radius $\sqrt{R^2 - t^2}$. Whence, if we define a continuous mapping on $\mathbf{R}^n$ by

$$\omega(x_1, x_2, \cdots, x_n) = \varsigma t(x_1, x_2, \cdots, x_n)$$

and

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n),$$

then we get easily an n-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq \frac{R^2 - t^2}{\varsigma^2 t^2},$$

of $\mathcal{B}$ under p for parameter t, which is just a pseudo-face of $\mathcal{B}$ on parameter t by definition.

For the case of $n = 2$ and $\varsigma = \frac{1}{2}$, since its pseudo-face is a circle on a Euclid plane and $-R \leq t \leq R$, we get an elliptic ball as shown in Fig.10.1.1. □

Similarly, if we define $\omega(x_1, x_2, \cdots, x_n) = 2\angle(\overrightarrow{OP}, Ot)$ for a point $P = (x_1, x_2, \cdots, x_n, t)$, i.e., an angle function, we can also get a result like Theorem 10.1.2 for pseudo-shapes of an $(n+1)$-ball.

Theorem 10.1.4 *Let $\mathcal{B}$ be an $(n+1)$-ball of radius R in space $\mathbf{R}^{n+1}$, i.e.,*

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \leq R^2.$$

Define a continuous mapping $\omega : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\omega(x_1, x_2, \cdots, x_n) = 2\angle(\overrightarrow{OP}, Ot)$$

for a point P on $\mathcal{B}$ and a continuous mapping $p : \mathbf{R}^{n+1} \to \mathbf{R}^n$ by

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n).$$

Then the pseudo-shape of $\mathcal{B}$ in $(\mathbf{R}^n, \omega)$ is a ball of radius $\sqrt{R^2 - t^2}$ for any parameter $t, -R \leq t \leq R$. Particularly, if $n = 2$, it is a circle of radius $\sqrt{R^2 - t^2}$ on parameter t and a body in $\mathbf{R}^3$ with equations

$$\oint \arctan(\frac{t}{x}) = 2\pi \ \ \textit{and} \ \ \oint \arctan(\frac{t}{y}) = 2\pi$$

for curves of its intersection with planes XOT and YOT.

Proof The proof is similar to that of Theorem 10.1.3, and these equations

$$\oint \arctan(\frac{t}{x}) = 2\pi \ \ \text{or} \ \ \oint \arctan(\frac{t}{y}) = 2\pi$$

are implied by the geometrical meaning of an angle function in the case of $n = 2$. □

10.1.3 Subspace Inclusion. For a Euclid space $\mathbf{R}^n$, we can get a subspace sequence

$$\mathbf{R}_0 \supset \mathbf{R}_1 \supset \cdots \supset \mathbf{R}_{n-1} \supset \mathbf{R}_n,$$

where the dimension of $\mathbf{R}_i$ is $n - i$ for $1 \leq i \leq n$ and $\mathbf{R}_n$ is just a point. Generally, we can not get a sequence in a reversing order, i.e., a sequence

$$\mathbf{R}_0 \subset \mathbf{R}_1 \subset \cdots \subset \mathbf{R}_{n-1} \subset \mathbf{R}_n$$

in classical space theory. By applying Smarandache multi-spaces, we can really find this kind of sequence, which can be used to explain a well-known model for our cosmos in M-theory.

Theorem 10.1.5 *Let $P = (x_1, x_2, \cdots, x_n)$ be a point in $\mathbf{R}^n$. Then there are subspaces of dimensional s in P for any integer $s, 1 \leq s \leq n$.*

Proof Notice that there is a normal basis $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ (every entry is 0 unless the i-th entry is 1), $\cdots$, $e_n = (0, 0, \cdots, 0, 1)$ in a Euclid space $\mathbf{R}^n$ such that

$$(x_1, x_2, \cdots, x_n) = x_1e_1 + x_2e_2 + \cdots + x_ne_n$$

for any point $(x_1, x_2, \cdots, x_n)$ in $\mathbf{R}^n$. Now consider a linear space $\mathbf{R}^- = (V, +_{new}, \circ_{new})$ on a field $F = \{a_i, b_i, c_i, \cdots, d_i; i \geq 1\}$, where $V = \{x_1, x_2, \cdots, x_n\}$. Not loss of generality, we assume that $x_1, x_2, \cdots, x_s$ are independent, i.e., if there exist scalars $a_1, a_2, \cdots, a_s$ such that

$$a_1 \circ_{new} x_1 +_{new} a_2 \circ_{new} x_2 +_{new} \cdots +_{new} a_s \circ_{new} x_s = 0,$$

then $a_1 = a_2 = \cdots = 0_{new}$ and there are scalars $b_i, c_i, \cdots, d_i$ with $1 \leq i \leq s$ in $\mathbf{R}^-$ such that

$$x_{s+1} = b_1 \circ_{new} x_1 +_{new} b_2 \circ_{new} x_2 +_{new} \cdots +_{new} b_s \circ_{new} x_s;$$

$$x_{s+2} = c_1 \circ_{new} x_1 +_{new} c_2 \circ_{new} x_2 +_{new} \cdots +_{new} c_s \circ_{new} x_s;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$x_n = d_1 \circ_{new} x_1 +_{new} d_2 \circ_{new} x_2 +_{new} \cdots +_{new} d_s \circ_{new} x_s.$$

Consequently, we get a subspace of dimensional s in point P of $\mathbf{R}^n$. □

Corollary 10.1.3 *Let P be a point in a Euclid space $\mathbf{R}^n$. Then there is a subspace sequence*

$$\mathbf{R}_0^- \subset \mathbf{R}_1^- \subset \cdots \subset \mathbf{R}_{n-1}^- \subset \mathbf{R}_n^-$$

such that $\mathbf{R}_n^- = \{P\}$ and the dimension of the subspace $\mathbf{R}_i^-$ is $n - i$, where $1 \leq i \leq n$.

Proof Applying Theorem 10.1.5 repeatedly, we can get such a sequence. □

§10.2 RELATIVITY THEORY

10.2.1 Spacetime. In theoretical physics, these spacetimes are used to describe various states of particles dependent on the time in a Euclid space $\mathbf{R}^3$. There are two kinds of spacetimes. One is the *absolute spacetime* consisting of a Euclid space $\mathbf{R}^3$ and an independent time, denoted by $(x_1, x_2, x_3|t)$. Another is the *relative spacetime*, i.e., a Euclid space $\mathbf{R}^4$, where time is the t-axis, seeing also in [Car1] for details.

A point in a spacetime is called an *event*, i.e., represented by $(x_1, x_2, x_3) \in \mathbf{R}^3$ and $t \in \mathbf{R}^+$ in an absolute spacetime in Newton's mechanics and $(x_1, x_2, x_3, t) \in \mathbf{R}^4$ with time parameter t in a relative spacetime in Einstein's relativity theory.

For two events $A_1 = (x_1, x_2, x_3|t_1)$ and $A_2 = (y_1, y_2, y_3|t_2)$, the *time interval* $\triangle t$ is defined by $\triangle t = t_1 - t_2$ and the *space interval* $\triangle(A_1, A_2)$ by

$$\triangle(A_1, A_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Similarly, for two events $B_1 = (x_1, x_2, x_3, t_1)$ and $B_2 = (y_1, y_2, y_3, t_2)$, the *spacetime interval* $\triangle s$ is defined by

$$\triangle^2 s = -c^2 \triangle t^2 + \triangle^2(B_1, B_2),$$

where c is the speed of the light in vacuum. For example, a spacetime only with two parameters x, y and the time parameter t is shown in Fig.10.2.1.

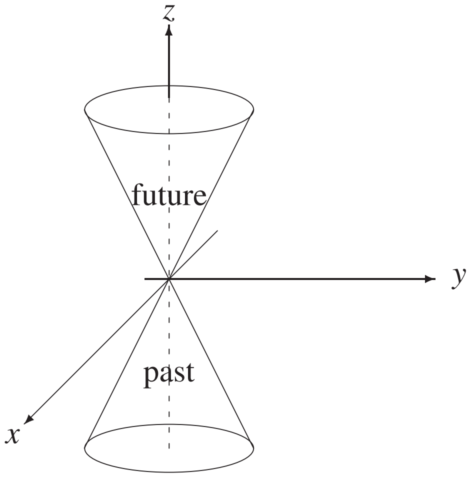


Fig.10.2.1

10.2.2 Lorentz Transformation. The Einstein's spacetime is a uniform linear space. By the assumption of linearity of spacetime and invariance of the light speed, it can be shown that the invariance of space-time intervals, i.e.,

For two reference systems S_1 and S_2 with a homogenous relative velocity, there must be

$$\triangle s^2 = \triangle s'^2.$$

We can also get the Lorentz transformation of spacetime or velocities by this assumption. For two parallel reference systems S_1 and S_2, if the velocity of S_2 relative to

S_1 along x-axis is v, such as shown in Fig.10.2.2,

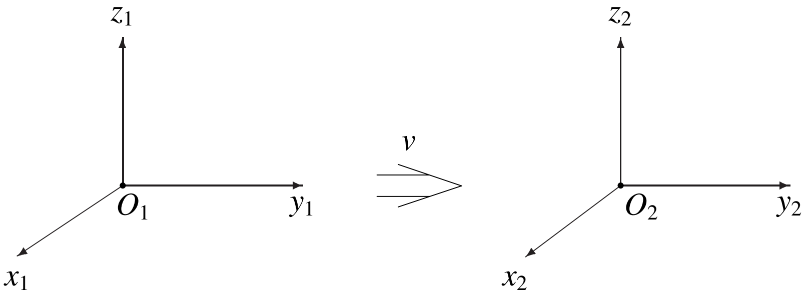


Fig.10.2.2

then the *Lorentz transformation of spacetime*, *transformation of velocity* are respectively

$$\left\{\begin{array}{l} x_2 = \dfrac{x_1 - vt_1}{\sqrt{1-(\frac{v}{c})^2}} \\ y_2 = y_1 \\ z_2 = z_1 \\ t_2 = \dfrac{t_1 - \frac{v}{c}x_1}{\sqrt{1-(\frac{v}{c})^2}} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} v_{x_2} = \dfrac{v_{x_1} - v}{1-\frac{vv_{x_1}}{c^2}} \\ v_{y_2} = \dfrac{v_{y_1}\sqrt{1-(\frac{v}{c})^2}}{1-\frac{vv_{x_1}}{c^2}} \\ v_{z_2} = \dfrac{v_{z_1}\sqrt{1-(\frac{v}{c})^2}}{1-\frac{vv_{x_1}}{c^2}}. \end{array}\right.$$

In a relative spacetime, the *general interval* is defined by

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu},$$

where $g_{\mu\nu} = g_{\mu\nu}(x^{\sigma}, t)$ is a metric both dependent on the space and time. We can also introduce the invariance of general intervals, i.e.,

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = g'_{\mu\nu}dx'^{\mu}dx'^{\nu}.$$

Then the *Einstein's equivalence principle* says that

There are no difference for physical effects of the inertial force and the gravitation in a field small enough.

An immediately consequence of the this equivalence principle is the idea that the *geometrization of gravitation*, i.e., considering the curvature at each point in a spacetime to be all effect of gravitation), which is called a *gravitational factor* at that point.

Combining these discussions in Section 10.1.1 with Einstein's idea of the geometrization of gravitation, we get a result for spacetimes following.

Theorem 10.2.1 *Every spacetime is a pseudo-face in a Euclid pseudo-space, especially, the Einstein's space-time is* $\mathbf{R}^n$ *in* $(\mathbf{R}^4, \omega)$ *for an integer* $n, n \geq 4$.

By the uniformity of spacetime, we get an equation by equilibrium of vectors in cosmos following.

Theorem 10.2.2 *For a spacetime in* $(\mathbf{R}^4, \omega)$*, there exists an anti-vector* ω_O^- *of* ω_O *along any orientation* $\overrightarrow{O}$ *in* $\mathbf{R}^4$ *such that*

$$\omega_O + \omega_O^- = 0.$$

Proof Since $\mathbf{R}^4$ is uniformity, by the principle of equilibrium in uniform spaces, along any orientation $\overrightarrow{O}$ in $\mathbf{R}^4$, there must exists an anti-vector ω_O^- of ω_O such that

$$\omega_O + \omega_O^- = 0. \qquad \square$$

10.2.3 Einstein Gravitational Field. For a gravitational field, let

$$\omega_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu}$$

in Theorem 10.2.2. Then we get that

$$\omega_{\mu\nu}^- = -8\pi G T_{\mu\nu}.$$

Consequently, we get the *Einstein's equations*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

of gravitational field. For solving these equations, two assumptions following are needed. One is partially adopted from that Einstein's, another is suggested by ours.

Postulate 10.2.1 *At the beginning our cosmos is homogenous.*

Postulate 6.2.2 *Human beings can only survey pseudo-faces of our cosmos by observations and experiments.*

10.2.4 Schwarzschild Spacetime. A *Schwarzschild metric* is a spherically symmetric Riemannian metric

$$d^2s = g_{\mu\nu}dx^{\mu\nu}$$

used to describe the solution of Einstein gravitational field equations in vacuum due to a spherically symmetric distribution of matter. Usually, the coordinates for such space can be chosen to be the spherical coordinates (r, θ, ϕ), and consequently (t, r, θ, ϕ) the coordinates of a spherically symmetric spacetime. Then a standard such metric can be written as follows:

$$ds^2 = B(r,t)dt^2 - A(r,t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Solving these equations enables one to get the line element

$$ds^2 = f(t)\left(1 - \frac{r_g}{r}\right)dt^2 - \frac{1}{1 - \frac{r_g}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for Schwarzschild spaces. See [Car1] or [Mao36] for details.

The *Schwarzschild radius* r_s is defined to be

$$r_s = \frac{r_g}{c^2} = \frac{2Gm}{c^2}.$$

At its surface $r = r_s$, these metric tensors g_{rr} diverge and g_{tt} vanishes, which giving the existence of a singularity in Schwarzschild spacetime.

One can show that each line with constants t, θ and ϕ are geodesic lines. These geodesic lines are spacelike if $r > r_s$ and timelike if $r < r_s$. But the tangent vector of a geodesic line undergoes a parallel transport along this line and can not change from timelike to spacelike. Whence, the two regions $r > r_s$ and $r < r_s$ can not join smoothly at the surface $r = r_s$.

We can also find this fact if we examine the radical null directions along $d\theta = \phi = 0$. In such a case, we have

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 = 0.$$

Therefore, the radical null directions must satisfy the following equation

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_s}{r}\right)$$

in units in which the speed of light is unity. Notice that the timelike directions are contained within the light cone, we know that in the region $r > r_s$ the opening of light cone decreases with r and tends to 0 at $r = r_s$, such as those shown in Fig.10.2.3.

In the region $r < r_s$ the parametric lines of the time t become spacelike. Consequently, the light cones rotate 90°, such as those shown in Fig.10.2.3, and their openings increase when moving from $r = 0$ to $r = r_s$. Comparing the light cones on both sides of

$r = r_s$, we can easy find that these regions on the two sides of the surface $r = r_s$ do not join smoothly at $r = r_s$.

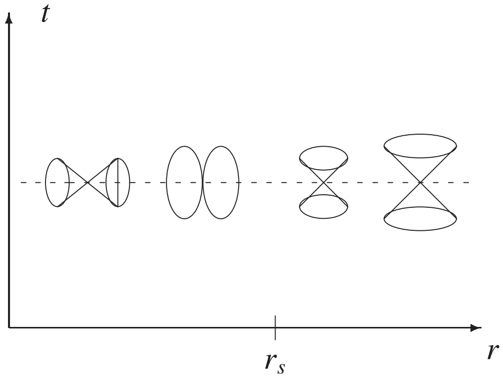


Fig. 10.2.3

10.2.5 Kruskal Coordinate. For removing the singularity appeared in Schwarzschild spacetime, Kruskal introduced a new spherically symmetric coordinate system, in which radical light rays have the slope $dr/dt = \pm 1$ everywhere. Then the line element will has a form

$$ds^2 = f^2 dt^2 - f^2 dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

By requiring the function f to depend only on r and to remain finite and nonzero for $u = v = 0$, we find a transformation between the exterior of the *spherically singularity* $r > r_s$ and the quadrant $u > |v|$ with new variables following:

$$v = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \sinh\left(\frac{t}{2r_s}\right),$$

$$u = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} \exp\left(\frac{r}{2r_s}\right) \cosh\left(\frac{t}{2r_s}\right).$$

The inverse transformations are given by

$$\left(\frac{r}{r_s} - 1\right) \exp\left(\frac{r}{2r_s}\right) = u^2 - v^2,$$

$$\frac{t}{2r_s} = \operatorname{arctanh}\left(\frac{v}{u}\right)$$

and the function f is defined by

$$\begin{aligned} f^2 &= \frac{32Gm^3}{r} \exp\left(-\frac{r}{r_s}\right) \\ &= \text{a transcendental function of } u^2 - v^2. \end{aligned}$$

This new coordinates present an analytic extension E of the limited region S of the Schwarzschild spacetime without singularity for $r > r_s$. The metric in the extended region joins on smoothly and without singularity to the metric at the boundary of S at $r = r_s$. This fact may be seen by a direction examination of the geodesics, i.e., every geodesic followed in which ever direction, either runs into the *barrier* of intrinsic singularity at $r = 0$, i.e., $v^2 - u^2 = 1$, or is continuable infinitely. Notice that this transformation also presents a *bridge* between two otherwise Euclidean spaces in topology, which can be interpreted as the *throat of a wormhole* connecting two distant regions in a Euclidean space.

10.2.6 Friedmann Cosmos. Applying these postulates, Einstein's gravitational equations and the *cosmological principle*, i.e., *there are no difference at different points and different orientations at a point of a cosmos on the metric* $10^4 l.y.$, we can get a standard model for cosmos, called the *Friedmann cosmos* by letting

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$

in Schwarzschild cosmos, seeing [Car1] for details. Such cosmoses are classified into three types:

Static Cosmos: $da/dt = 0$;

Contracting Cosmos: $da/dt < 0$;

Expanding Cosmos: $da/dt > 0$.

By Einstein's view, our living cosmos is the static cosmos. That is why he added a cosmological constant λ in his equation of gravitational field. But unfortunately, our cosmos is an expanding cosmos found by Hubble in 1929.

§10.3 A COMBINATORIAL MODEL FOR COSMOS

As shown in Chapter 2, a graph with more than 2 vertices is itself a multi-space with different vertices, edges two by two. As an application, we consider such multi-spaces for physics in this section.

10.3.1 M-Theory. Today, we have know that all matter are made of atoms and sub-atomic particles, held together by four fundamental forces: *gravity, electro-magnetism,* *strong nuclear force* and *weak force*. Their features are partially explained by the *quantum theory* and the *relativity theory*. The former is a theory for the microcosm but the later is for the macrocosm. However, these two theories do not resemble each other in any way. The quantum theory reduces forces to the exchange of discrete packet of quanta, while the relativity theory explains the cosmic forces by postulating the smooth deformation of the fabric spacetime.

As we known, there are two string theories : the $E_8 \times E_8$ heterotic string, the *SO(32) heterotic string* and three superstring theories: the *SO(32) Type I string*, the *Type IIA* and *Type IIB* in superstring theories. Two physical theories are *dual* to each other if they have identical physics after a certain mathematical transformation. There are *T-duality* and *S-duality* in superstring theories defined in the following table 10.3.1([Duf1]).

	fundamental string	dual string
T-duality	Radius↔ 1/(radius) Kaluza-Klein↔ Winding	charge↔ 1/(charge) Electric↔ Magnet
S-duality	charge↔ 1/(charge) Electric↔ Magnetic	Radius↔ 1/(Radius) Kaluza-Klein↔ Winding

table 10.3.1

We already know some profound properties for these spring or superspring theories, such as:

(1) Type *IIA* and *IIB* are related by T-duality, as are the two heterotic theories.

(2) Type *I* and heterotic $SO(32)$ are related by S-duality and Type *IIB* is also S-dual with itself.

(3) Type *II* theories have two supersymmetries in the 10-dimensional sense, but the rest just one.

(4) Type *I* theory is special in that it is based on unoriented open and closed strings, but the other four are based on oriented closed strings.

(5) The *IIA* theory is special because it is non-chiral(parity conserving), but the other four are chiral(parity violating).

(6) In each of these cases there is an 11th dimension that becomes large at strong coupling. For substance, in the *IIA* case the 11th dimension is a circle and in *IIB* case it is a line interval, which makes 11-dimensional spacetime display two 10-dimensional boundaries.

(7) The strong coupling limit of either theory produces an 11-dimensional space-time.

(8) $\cdots$, etc..

The M-theory was established by Witten in 1995 for the unity of those two string theories and three superstring theories, which postulates that all matter and energy can be reduced to *branes* of energy vibrating in an 11 dimensional space. This theory gives one a compelling explanation of the origin of our cosmos and combines all of existed string theories by showing those are just special cases of M-theory, such as those shown in the following.

$$M - theory \supset \begin{cases} E_8 \times E_8 \ heterotic\ string \\ SO(32)\ heterotic\ string \\ SO(32)\ Type\ I\ string \\ Type\ IIA \\ Type\ IIB. \end{cases}$$

See Fig.10.3.1 for the M-theory planet in which we can find a relation of M-theory with these two strings or three superstring theories.

The M-theory Planet

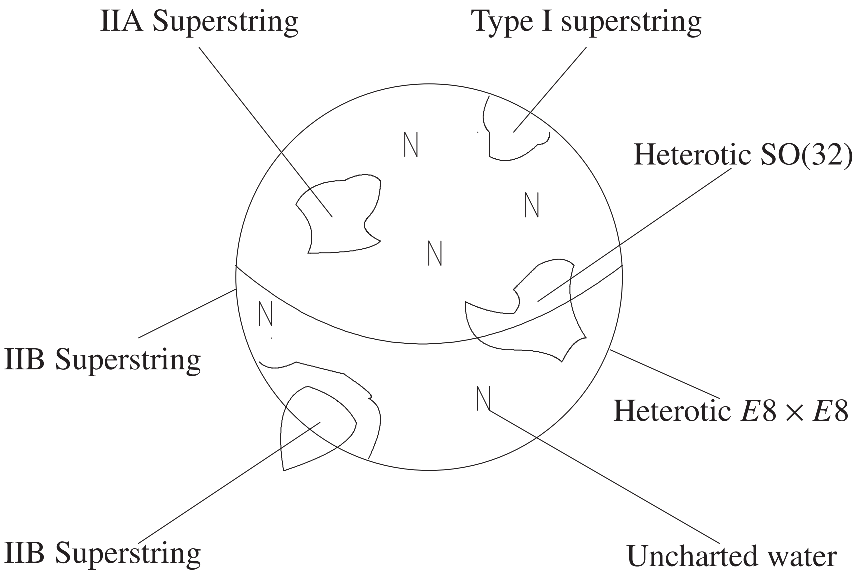


Fig.10.3.1

A widely accepted opinion on our cosmos is that itis in accelerating expansion, i.e., it is most possible an accelerating cosmos of expansion. This observation implies that it should satisfies the following condition

$$\frac{d^2a}{dt^2} > 0.$$

The *Kasner* type metric

$$ds^2 = -dt^2 + a(t)^2 d^2_{\mathbf{R}^3} + b(t)^2 ds^2(T^m)$$

solves the $4 + m$ dimensional vacuum Einstein equations if

$$a(t) = t^{\mu} \text{ and } b(t) = t\nu$$

with

$$\mu = \frac{3 \pm \sqrt{3m(m+2)}}{3(m+3)}, \nu = \frac{3 \mp \sqrt{3m(m+2)}}{3(m+3)}.$$

These solutions in general do not give an accelerating expansion of spacetime of dimension 4. However, by applying the time-shift symmetry

$$t \longrightarrow t_{+\infty} - t, \ \ a(t) = (t_{+\infty} - t)^{\mu},$$

we see that yields a really accelerating expansion since

$$\frac{da(t)}{dt} > 0 \text{ and } \frac{d^2a(t)}{dt^2} > 0.$$

According to M-theory, our cosmos started as a perfect 11 dimensional space with nothing in it. However, this 11 dimensional space was unstable. The original 11 dimensional spacetime finally cracked into two pieces, a 4 and a 7 dimensional cosmos. The cosmos made the 7 of the 11 dimensions curled into a tiny ball, allowing the remaining 4 dimensional cosmos to inflate at enormous rates. This originality of our cosmos implies a multi-space result for our cosmos verified by Theorem 10.1.5.

Theorem 10.3.1 *The spacetime of M-theory is a multi-space with a warping* $\mathbf{R}^7$ *at each point of* $\mathbf{R}^4$.

Applying Theorem 10.3.1, an example for an accelerating expansion cosmos of 4-dimensional cosmos from supergravity compactification on hyperbolic spaces is the *Townsend-Wohlfarth type* in which the solution is

$$ds^2 = e^{-m\phi(t)}\left(-S^6 dt^2 + S^2 dx_3^2\right) + r_C^2 e^{2\phi(t)} ds^2_{H_m},$$

where

$$\phi(t) = \frac{1}{m-1}(\ln K(t) - 3\lambda_0 t), \quad S^2 = K^{\frac{m}{m-1}} e^{-\frac{m+2}{m-1}\lambda_0 t}$$

and

$$K(t) = \frac{\lambda_0 \zeta r_c}{(m-1)\sin[\lambda_0 \zeta |t + t_1|]}$$

with $\zeta = \sqrt{3 + 6/m}$. This solution is obtainable from space-like brane solution and if the proper time ς is defined by $d\varsigma = S^3(t)dt$, then the conditions for expansion and acceleration are $\frac{dS}{d\varsigma} > 0$ and $\frac{d^2S}{d\varsigma^2} > 0$. For example, the expansion factor is 3.04 if $m = 7$, i.e., a really expanding cosmos.

10.3.2 Pseudo-Face Model of *p*-Brane. In fact, M-theory contains much more than just strings, which is also implied in Fig.10.3.1. It contains both higher and lower dimensional objects, called *branes*. A *brane* is an object or subspace which can have various spatial dimensions. For any integer $p \geq 0$, a *p-brane* has length in p dimensions, for example, a 0-*brane* is just a point; a 1-*brane* is a string and a 2-*brane* is a surface or membrane $\cdots$.

For example, two branes and their motion have been shown in Fig.10.3.2, where (a) is a 1-brane and (b) is a 2-brane.

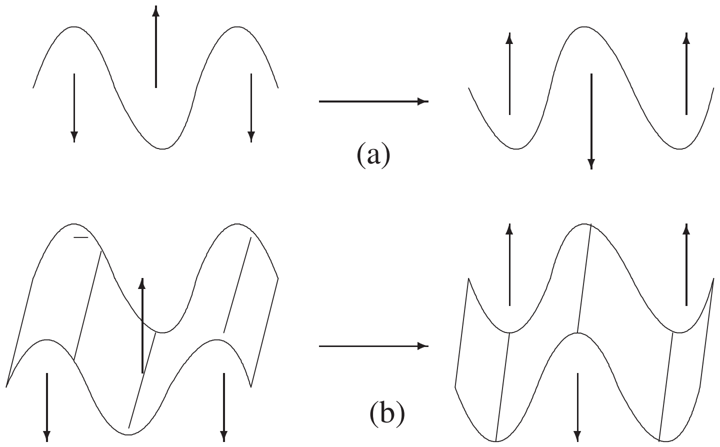


Fig.10.3.2

Combining these ideas in the pseudo-spaces theory and M-theory, a model for $\mathbf{R}^m$ by combinatorial manifolds is constructed in the below.

Model 10.3.1 *For each m-brane* $\mathbf{B}$ *of a space* $\mathbf{R}^m$, *let* $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ *be its unit vibrating normal vector along these p directions and* $q : \mathbf{R}^m \to \mathbf{R}^4$ *a continuous mapping.*

Now for $\forall P \in \mathbf{B}$*, define*

$$\omega(q(P)) = (n_1(P), n_2(P), \cdots, n_p(P)).$$

Then $(\mathbf{R}^4, \omega)$ *is a pseudo-face of* $\mathbf{R}^m$*, particularly, if* $m = 11$*, it is a pseudo-face for the M-theory.*

If $p = 4$, the interesting conclusions are obtained by applying results in Chapters 9.

Theorem 10.3.2 *For a sphere-like cosmos* $\mathbf{B}^2$*, there is a continuous mapping* $q : \mathbf{B}^2 \to \mathbf{R}^2$ *such that its spacetime is a pseudo-plane.*

Proof According to the classical geometry, we know that there is a projection $q : \mathbf{B}^2 \to \mathbf{R}^2$ from a 2-ball $\mathbf{B}^2$ to a Euclid plane $\mathbf{R}^2$, as shown in Fig.10.3.3.

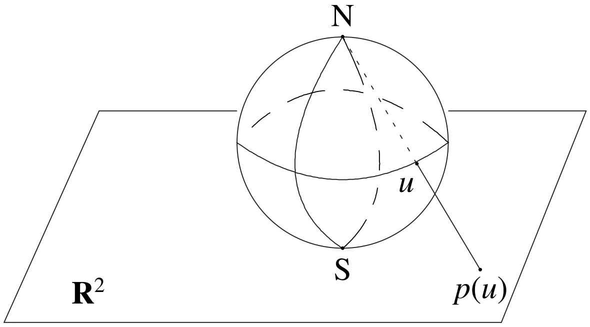


Fig.10.3.3

Now for any point $u \in \mathbf{B}^2$ with an unit vibrating normal vector $(x(u), y(u), z(u))$, define

$$\omega(q(u)) = (z(u), t),$$

where t is the time parameter. Then $(\mathbf{R}^2, \omega)$ is a pseudo-face of $(\mathbf{B}^2, t)$. $\square$

Generally, we can also find pseudo-surfaces as a pseudo-face of sphere-like cosmoses.

Theorem 10.3.3 *For a sphere-like cosmos* $\mathbf{B}^2$ *and a surface* S*, there is a continuous mapping* $q : \mathbf{B}^2 \to S$ *such that its spacetime is a pseudo-surface on* S*.*

Proof According to the classification theorem of surfaces, an surface S can be combinatorially represented by a $2n$-polygon for an integer $n, n \geq 1$. If we assume that each edge of this polygon is at an infinite place, then the projection in Fig.6.6 also enables us to get a continuous mapping $q : \mathbf{B}^2 \to S$. Thereby we get a pseudo-face on S for the cosmos $\mathbf{B}^2$. $\square$

Furthermore, we can construct a combinatorial model for our cosmos.

Model 10.3.2 *For each m-brane* $\mathbf{B}$ *of a space* $\mathbf{R}^m$*, let* $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ *be its unit vibrating normal vector along these p directions and* $q : \mathbf{R}^m \to \mathbf{R}^4$ *a continuous mapping. Now construct a graph phase* $(\mathcal{G}, \omega, \Lambda)$ *by*

$$V(\mathcal{G}) = \{p - branes\ q(\mathbf{B})\},$$

$$E(\mathcal{G}) = \{(q(\mathbf{B}_1), q(\mathbf{B}_2)) | there\ is\ an\ action\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2\},$$

$$\omega(q(\mathbf{B})) = (n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B})),$$

and

$$\Lambda(q(\mathbf{B}_1), q(\mathbf{B}_2)) = \ forces\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2.$$

Then we get a graph phase $(\mathcal{G}, \omega, \Lambda)$ *in* $\mathbf{R}^4$*. Similarly, if* $m = 11$*, it is a graph phase for the M-theory.*

If there are only finite p-branes in our cosmos, then Theorems 10.3.2 and 10.3.3 can be restated as follows.

Theorem 10.3.4 *For a sphere-like cosmos* $\mathbf{B}^2$ *with finite p-branes and a surface* S*, its spacetime is a map geometry on* S*.*

Now we consider the transport of a graph phase $(\mathcal{G}, \omega, \Lambda)$ in $\mathbf{R}^m$ by applying conclusions in Chapter 2.

Theorem 10.3.5 *A graph phase* $(\mathcal{G}_1, \omega_1, \Lambda_1)$ *of space* $\mathbf{R}^m$ *is transformable to a graph phase* $(\mathcal{G}_2, \omega_2, \Lambda_2)$ *of space* $\mathbf{R}^n$ *if and only if* $\mathcal{G}_1$ *is embeddable in* $\mathbf{R}^n$ *and there is a continuous mapping* τ *such that* $\omega_2 = \tau(\omega_1)$ *and* $\Lambda_2 = \tau(\Lambda_1)$*.*

Proof If $(\mathcal{G}_1, \omega_1, \Lambda_1)$ is transformable to $(\mathcal{G}_2, \omega_2, \Lambda_2)$, by the definition of transformation there must be $\mathcal{G}_1$ embeddable in $\mathbf{R}^n$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$ and $\Lambda_2 = \tau(\Lambda_1)$.

Now if $\mathcal{G}_1$ is embeddable in $\mathbf{R}^n$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$, $\Lambda_2 = \tau(\Lambda_1)$, let $\varsigma : \mathcal{G}_1 \to \mathcal{G}_2$ be a continuous mapping from $\mathcal{G}_1$ to $\mathcal{G}_2$, then (ς, τ) is continuous and

$$(\varsigma, \tau) : (\mathcal{G}_1, \omega_1, \Lambda_1) \to (\mathcal{G}_2, \omega_2, \Lambda_2).$$

Therefore $(\mathcal{G}_1, \omega_1, \Lambda_1)$ is transformable to $(\mathcal{G}_2, \omega_2, \Lambda_2)$. □

Theorem 10.3.5 has many interesting consequences as by-products.

Corollary 10.3.1 *A graph phase* $(\mathcal{G}_1, \omega_1, \Lambda_1)$ *in* $\mathbf{R}^m$ *is transformable to a planar graph phase* $(\mathcal{G}_2, \omega_2, \Lambda_2)$ *if and only if* $\mathcal{G}_2$ *is a planar embedding of* $\mathcal{G}_1$ *and there is a continuous mapping* τ *such that* $\omega_2 = \tau(\omega_1)$, $\Lambda_2 = \tau(\Lambda_1)$ *and vice via, a planar graph phase* $(\mathcal{G}_2, \omega_2, \Lambda_2)$ *is transformable to a graph phase* $(\mathcal{G}_1, \omega_1, \Lambda_1)$ *in* $\mathbf{R}^m$ *if and only if* $\mathcal{G}_1$ *is an embedding of* $\mathcal{G}_2$ *in* $\mathbf{R}^m$ *and there is a continuous mapping* τ^{-1} *such that* $\omega_1 = \tau^{-1}(\omega_2)$, $\Lambda_1 = \tau^{-1}(\Lambda_2)$.

Corollary 10.3.2 *For a continuous mapping* τ, *a graph phase* $(\mathcal{G}_1, \omega_1, \Lambda_1)$ *in* $\mathbf{R}^m$ *is transformable to a graph phase* $(\mathcal{G}_2, \tau(\omega_1), \tau(\Lambda_1))$ *in* $\mathbf{R}^n$ *with* $m, n \geq 3$.

Proof This result follows immediately from Theorems 5.2.2 and 10.3.5. □

Theorem 10.3.5 can be also used to explain the problems of *travelling between cosmoses* or *getting into the heaven or hell* for a person. We all know that water can go from liquid phase to steam phase by heating and then come back to liquid phase by cooling because its phase is transformable between liquid phase and steam phase. Thus it satisfies the conditions of Theorem 10.3.5. For a person on the earth, he can only get into the heaven or hell after death because the dimensions of the heaven and that of the hell are respectively more or less than 4 and there are no transformations from a pattern of alive person in cosmos to that of in heaven or hell by the biological structure of his body. Whence, if the black holes are really these tunnels between different cosmoses, the destiny for a cosmonaut unfortunately fell into a black hole is only the death ([Haw1]-[Haw3]). Perhaps, there are other kind of beings found by human beings in the future who can freely change his phase from one state in space $\mathbf{R}^m$ to another in $\mathbf{R}^n$ with $m > n$ or $m < n$. Then at that time, the travelling between cosmoses is possible for those beings.

10.3.3 Combinatorial Cosmos. Until today, many problems in cosmology are puzzling one's eyes. Comparing with these vast cosmoses, human beings are very tiny. In spite of this depressed fact, we can still investigate cosmoses by our deeply thinking. Motivated by this belief, a multi-space model for cosmoses, called combinatorial cosmos is introduced following.

Model 10.3.3 *A combinatorial cosmos is constructed by a triple* (Ω, Δ, T)*, where*

$$\Omega = \bigcup_{i\geq 0} \Omega_i, \quad \Delta = \bigcup_{i\geq 0} O_i$$

and $T = \{t_i; i \geq 0\}$ *are respectively called the cosmos, the operation or the time set with the following conditions hold.*

(1) (Ω, Δ) *is a Smarandache multi-space dependent on* T*, i.e., the cosmos* (Ω_i, O_i) *is dependent on time parameter* t_i *for any integer* $i, i \geq 0$.

(2) *For any integer* $i, i \geq 0$*, there is a sub-cosmos sequence*

$$(S): \ \Omega_i \supset \cdots \supset \Omega_{i1} \supset \Omega_{i0}$$

in the cosmos (Ω_i, O_i) *and for two sub-cosmoses* (Ω_{ij}, O_i) *and* (Ω_{il}, O_i)*, if* $\Omega_{ij} \supset \Omega_{il}$*, then there is a homomorphism* $\rho_{\Omega_{ij},\Omega_{il}} : (\Omega_{ij}, O_i) \to (\Omega_{il}, O_i)$ *such that*

(i) *for* $\forall(\Omega_{i1}, O_i), (\Omega_{i2}, O_i), (\Omega_{i3}, O_i) \in (S)$*, if* $\Omega_{i1} \supset \Omega_{i2} \supset \Omega_{i3}$*, then*

$$\rho_{\Omega_{i1},\Omega_{i3}} = \rho_{\Omega_{i1},\Omega_{i2}} \circ \rho_{\Omega_{i2},\Omega_{i3}},$$

where "$\circ$*" denotes the composition operation on homomorphisms.*

(ii) *for* $\forall g, h \in \Omega_i$*, if for any integer* i*,* $\rho_{\Omega,\Omega_i}(g) = \rho_{\Omega,\Omega_i}(h)$*, then* $g = h$.

(iii) *for* $\forall i$*, if there is an* $f_i \in \Omega_i$ *with*

$$\rho_{\Omega_i,\Omega_i \bigcap \Omega_j}(f_i) = \rho_{\Omega_j,\Omega_i \bigcap \Omega_j}(f_j)$$

for integers $i, j, \Omega_i \bigcap \Omega_j \neq \emptyset$*, then there exists an* $f \in \Omega$ *such that* $\rho_{\Omega,\Omega_i}(f) = f_i$ *for any integer* i.

These conditions in (2) are used to ensure that a combinatorial cosmos posses the *general structure sheaf* of topological space, for instance if we equip each multi-space (Ω_i, O_i) with an Abelian group G_i for any integer $i, i \geq 0$, then we get structure sheaf on a combinatorial cosmos. This structure enables that a being in a cosmos of higher dimension can supervises those in lower dimension. For *general sheaf theory*, the reader is referred to the reference [Har1] for details.

By Model 10.3.3, there is just one cosmos Ω and the sub-cosmos sequence is

$$\mathbf{R}^4 \supset \mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\} \supset \mathbf{R}_7^- \supset \cdots \supset \mathbf{R}_1^- \supset \mathbf{R}_0^- = \{Q\}.$$

in the string/M-theory. In Fig.10.3.4, we have shown the idea of the combinatorial cosmos.

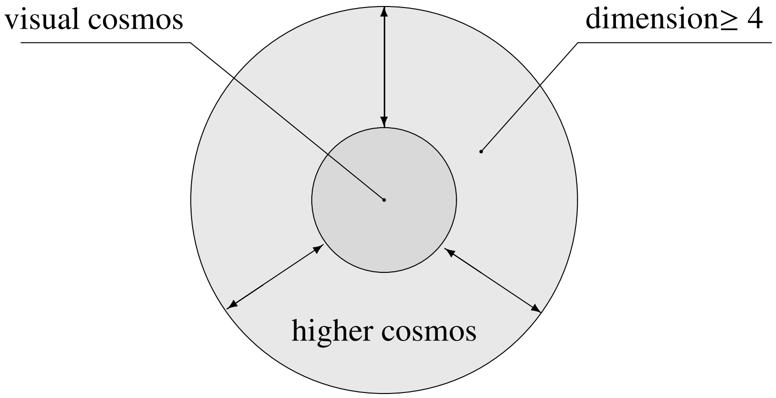


Fig.10.3.4

For 5 or 6 dimensional spaces, it has been established a dynamical theory by this combinatorial speculation([Pap1]-[Pap2]). In this dynamics, we look for a solution in the Einstein's equation of gravitational field in 6-dimensional spacetime with a metric of the form

$$ds^2 = -n^2(t,y,z)dt^2 + a^2(t,y,z)d\sum_k^2 + b^2(t,y,z)dy^2 + d^2(t,y,z)dz^2$$

where $d\sum_k^2$ represents the 3-dimensional spatial sections metric with $k = -1, 0, 1$ respective corresponding to the hyperbolic, flat and elliptic spaces. For 5-dimensional spacetime, deletes the indefinite z in this metric form. Now consider a 4-brane moving in a 6-dimensional *Schwarzschild-ADS spacetime*, the metric can be written as

$$ds^2 = -h(z)dt^2 + \frac{z^2}{l^2}d\sum_k^2 + h^{-1}(z)dz^2,$$

where

$$d\sum_k^2 = \frac{dr^2}{1-kr^2} + r^2d\Omega_{(2)}^2 + (1-kr^2)dy^2$$

and

$$h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3}.$$

Then the equation of a 4-dimensional cosmos moving in a 6-spacetime is

$$2\frac{\ddot{R}}{R} + 3(\frac{\dot{R}}{R})^2 = -3\frac{\kappa_{(6)}^4}{64}\rho^2 - \frac{\kappa_{(6)}^4}{8}\rho p - 3\frac{\kappa}{R^2} - \frac{5}{l^2}$$

by applying the *Darmois-Israel conditions* for a moving brane. Similarly, for the case of $a(z) \neq b(z)$, the equations of motion of the brane are

$$\frac{d^2\dot{d}\dot{R} - d\ddot{R}}{\sqrt{1+d^2\dot{R}^2}} - \frac{\sqrt{1+d^2\dot{R}^2}}{n}\left(d\dot{n}\dot{R} + \frac{\partial_z n}{d} - (d\partial_z n - n\partial_z d)\dot{R}^2\right) = -\frac{\kappa_{(6)}^4}{8}\left(3(p+\rho)+\hat{p}\right),$$

$$\frac{\partial_z a}{ad}\sqrt{1+d^2\dot{R}^2} = -\frac{\kappa_{(6)}^4}{8}(\rho + p - \hat{p}),$$

$$\frac{\partial_z b}{bd}\sqrt{1+d^2\dot{R}^2} = -\frac{\kappa_{(6)}^4}{8}(\rho - 3(p - \hat{p})),$$

where the energy-momentum tensor on the brane is

$$\hat{T}_{\mu\nu} = h_{\nu\alpha}T_\mu^\alpha - \frac{1}{4}Th_{\mu\nu}$$

with $T_\mu^\alpha = diag(-\rho, p, p, p, \hat{p})$ and the *Darmois-Israel conditions*

$$[K_{\mu\nu}] = -\kappa_{(6)}^2\hat{T}_{\mu\nu},$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor.

10.3.4 Combinatorial Gravitational Field. A *parallel probe* on a combinatorial Euclidean space $\widetilde{\mathbf{R}} = \bigcup\limits_{i=1}^{m} \mathbf{R}^{n_i}$ is the set of probes established on each Euclidean space $\mathbf{R}^{n_i}$ for integers $1 \leq i \leq m$, particularly for $\mathbf{R}^{n_i} = \mathbf{R}^3$ for integers $1 \leq i \leq m$ which one can detects a particle in its each space $\mathbf{R}^3$ such as those shown in Fig.10.3.5 in where $G = K_4$ and there are four probes P_1, P_2, P_3, P_4.

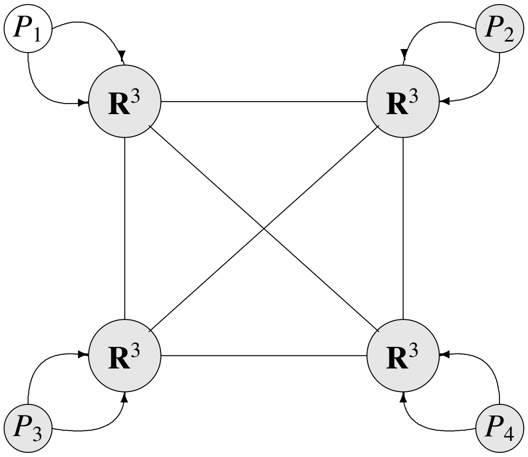


Fig.10.3.5

Notice that data obtained by such parallel probe is a set of local data $F(x_{i1}, x_{i2}, x_{i3})$ for $1 \leq i \leq m$ underlying G, i.e., the detecting data in a spatial $\overline{\epsilon}$ should be same if $\overline{\epsilon} \in \mathbf{R}^3_u \cap \mathbf{R}^3_v$, where $\mathbf{R}^3_u$ denotes the $\mathbf{R}^3$ at $u \in V(G)$ and $\left(\mathbf{R}^3_u, \mathbf{R}^3_v\right) \in E(G)$.

For data not in the $\mathbf{R}^3$ we lived, it is reasonable that we can conclude that all are the same as we obtained. Then we can analyze the global behavior of a particle in Euclidean space $\mathbf{R}^n$ with $n \geq 4$. Let us consider the gravitational field with dimensional≥ 4. We know the Einstein's gravitation field equations in $\mathbf{R}^3$ are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},$$

where $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$ are the respective *Ricci tensor*, *Ricci scalar curvature* and

$$\kappa = \frac{8\pi G}{c^4} = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$$

Now for a gravitational field $\mathbf{R}^n$ with $n \geq 4$, we decompose it into dimensional 3 Euclidean spaces $\mathbf{R}^3_u, \mathbf{R}^3_v, \cdots, \mathbf{R}^3_w$. Then we find Einstein's gravitational equations as follows:

$$R_{\mu_u\nu_u} - \frac{1}{2}g_{\mu_u\nu_u}R = -8\pi G\mathscr{E}_{\mu_u\nu_u},$$

$$R_{\mu_v\nu_v} - \frac{1}{2}g_{\mu_v\nu_v}R = -8\pi G\mathscr{E}_{\mu_v\nu_v},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$R_{\mu_w\nu_w} - \frac{1}{2}g_{\mu_w\nu_w}R = -8\pi G\mathscr{E}_{\mu_w\nu_w}$$

for each $\mathbf{R}^3_u$, $\mathbf{R}^3_v, \cdots,$ $\mathbf{R}^3_w$. If we decompose $\mathbf{R}^n$ into a combinatorial Euclidean fan-space $\widetilde{R}(\underbrace{3, 3, \cdots, 3}_{m})$, then $u, v, \cdots, w$ can be abbreviated to $1, 2\cdots, m$. In this case, these gravitational equations can be represented by

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2}g_{(\mu\nu)(\sigma\tau)}R = -8\pi G\mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

with a coordinate matrix

$$[\overline{x}_p] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & \cdots & x^{13} \\ x^{21} & \cdots & x^{2\widehat{m}} & \cdots & x^{23} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{m1} & \cdots & x^{m\widehat{m}} & \cdots & x^{m3} \end{bmatrix}$$

for a point $p \in \mathbf{R}^n$, where $\widehat{m} = \dim\left(\bigcap\limits_{i=1}^{m}\mathbf{R}^{n_i}\right)$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m}\mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. Because the local behavior is that of the projection of the global. Whence, the following principle for determining behavior of particles in $\mathbf{R}^n$, $n \geq 4$ hold.

Projective Principle *A physics law in a Euclidean space* $\mathbf{R}^n \simeq \widetilde{\mathbf{R}} = \bigcup\limits_{i=1}^{n}\mathbf{R}^3$ *with* $n \geq 4$ *is invariant under a projection on* $\mathbf{R}^3$ *in* $\widetilde{\mathbf{R}}$.

Applying this principe enables us to find a spherically symmetric solution of Einstein's gravitational equations in Euclidean space $\mathbf{R}^n$.

A *combinatorial metric* is defined by

$$ds^2 = g_{(\mu\nu)(\kappa\lambda)}dx^{\mu\nu}dx^{\kappa\lambda},$$

where $g_{(\mu\nu)(\kappa\lambda)}$ is the Riemannian metric in $\left(\widetilde{M}, g, \widetilde{D}\right)$. Generally, we can choose a orthogonal basis $\{\overline{e}_{11}, \cdots, \overline{e}_{1n_1}, \cdots, \overline{e}_{s(p)n_{s(p)}}\}$ for $\varphi_p[U]$, $p \in \widetilde{M}(t)$, i.e., $\left\langle\overline{e}_{\mu\nu}, \overline{e}_{\kappa\lambda}\right\rangle = \delta_{(\mu\nu)}^{(\kappa\lambda)}$. Then

$$\begin{aligned} ds^2 &= g_{(\mu\nu)(\mu\nu)}(dx^{\mu\nu})^2 \\ &= \sum_{\mu=1}^{s(p)}\sum_{\nu=1}^{\widehat{s}(p)} g_{(\mu\nu)(\mu\nu)}(dx^{\mu\nu})^2 + \sum_{\mu=1}^{s(p)}\sum_{\nu=1}^{\widehat{s}(p)+1} g_{(\mu\nu)(\mu\nu)}(dx^{\mu\nu})^2 \\ &= \frac{1}{s^2(p)}\sum_{\nu=1}^{\widehat{s}(p)}(\sum_{\mu=1}^{s(p)} g_{(\mu\nu)(\mu\nu)})dx^{\nu} + \sum_{\mu=1}^{s(p)}\sum_{\nu=1}^{\widehat{s}(p)+1} g_{(\mu\nu)(\mu\nu)}(dx^{\mu\nu})^2, \end{aligned}$$

which enables one find an important relation of combinatorial metric with that of its projections following.

Theorem 10.3.6 *Let* ${}_{\mu}ds^2$ *be the metric of* $\phi_p^{-1}(B^{n_\mu(p)})$ *for integers* $1 \leq \mu \leq s(p)$. *Then*

$$ds^2 = {}_1ds^2 + {}_2ds^2 + \cdots + {}_{s(p)}ds^2.$$

Proof Applying the projective principle, we immediately know that

$${}_{\mu}ds^2 = ds^2|_{\phi_p^{-1}(B^{n_\mu(p)})}, \qquad 1 \leq \mu \leq s(p).$$

Whence, we find that

$$\begin{aligned} ds^2 &= g_{(\mu\nu)(\mu\nu)}(dx^{\mu\nu})^2 = \sum_{\mu=1}^{s(p)}\sum_{\nu=1}^{n_i(p)} g_{(\mu\nu)(\mu\nu)}\,(dx^{\mu\nu})^2 \\ &= \sum_{\mu=1}^{s(p)} ds^2\Big|_{\phi_p^{-1}(B^{n_\mu(p)})} = \sum_{\mu=1}^{s(p)} {}_{\mu}ds^2. \end{aligned}$$

□

Let M be a gravitational field. We have known its Schwarzschild metric, i.e., a spherically symmetric solution of Einstein's gravitational equations in vacuum is

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2,$$

where $r_s = 2Gm/c^2$. Now we generalize it to combinatorial gravitational fields to find the solutions of equations

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2}g_{(\mu\nu)(\sigma\tau)}R = -8\pi G\mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

in vacuum, i.e., $\mathscr{E}_{(\mu\nu)(\sigma\tau)} = 0$. For such a objective, we only consider the homogenous combinatorial Euclidean spaces $\widetilde{M} = \bigcup_{i=1}^{m} \mathbf{R}^{n_i}$, i.e., for any point $p \in \widetilde{M}$,

$$[\varphi_p] = \left[\begin{array}{cccccccc} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & \cdots & x^{mn_m} \end{array}\right]$$

with $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$.

Let $\widetilde{M}(t)$ be a combinatorial field of gravitational fields $M_1, M_2, \cdots, M_m$ with masses $m_1, m_2, \cdots, m_m$ respectively. For usually undergoing, we consider the case of $n_\mu = 4$ for $1 \leq \mu \leq m$ since line elements have been found concretely in classical gravitational field in these cases. Now establish m spherical coordinate subframe $(t_\mu; r_\mu, \theta_\mu, \phi_\mu)$ with its originality at the center of such a mass space. Then we have known its a spherically symmetric solution to be

$$ds_\mu^2 = \left(1 - \frac{r_{\mu s}}{r_\mu}\right)dt_\mu^2 - \left(1 - \frac{r_{\mu s}}{r_\mu}\right)^{-1} dr_\mu^2 - r_\mu^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

for $1 \leq \mu \leq m$ with $r_{\mu s} = 2Gm_\mu/c^2$. By Theorem 8.3.1, we know that

$$ds^2 = {}_1ds^2 + {}_2ds^2 + \cdots + {}_mds^2,$$

where ${}_\mu ds^2 = ds_\mu^2$ by the projective principle on combinatorial fields. Notice that $1 \leq \widehat{m} \leq 4$. We therefore get combinatorial metrics dependent on $\widehat{m}$ following.

Case 1. *$\widehat{m} = 1$, i.e., $t_\mu = t$ for $1 \leq \mu \leq m$.*

In this case, the combinatorial metric ds is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)dt^2 - \sum_{\mu=1}^{m}(1 - \frac{2Gm_\mu}{c^2 r_\mu})^{-1}dr_\mu^2 - \sum_{\mu=1}^{m} r_\mu^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

Case 2. *$\widehat{m} = 2$, i.e., $t_\mu = t$ and $r_\mu = r$, or $t_\mu = t$ and $\theta_\mu = \theta$, or $t_\mu = t$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$.*

We consider the following subcases.

Subcase 2.1. *$t_\mu = t$, $r_\mu = r$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)dt^2 - (\sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r}\right)^{-1})dr^2 - \sum_{\mu=1}^{m} r^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2),$$

which can only happens if these m fields are at a same point O in a space. Particularly, if $m_\mu = M$ for $1 \leq \mu \leq m$, the masses of $M_1, M_2, \cdots, M_m$ are the same, then $r_{\mu g} = 2GM$ is a constant, which enables us knowing that

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)mdt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} mdr^2 - \sum_{\mu=1}^{m} r^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

Subcase 2.2. *$t_\mu = t$, $\theta_\mu = \theta$.*

In this subcase, the combinatorial metric is

$$\begin{aligned} ds^2 &= \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)dt^2 \\ &- \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)^{-1} dr_\mu^2 - \sum_{\mu=1}^{m} r_\mu^2(d\theta^2 + \sin^2\theta d\phi_\mu^2). \end{aligned}$$

Subcase 2.3. *$t_\mu = t$, $\phi_\mu = \phi$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)dt^2 - (\sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)^{-1})dr_\mu^2 - \sum_{\mu=1}^{m} r_\mu^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi^2).$$

Case 3. *$\widehat{m} = 3$, i.e., $t_\mu = t$, $r_\mu = r$ and $\theta_\mu = \theta$, or $t_\mu = t$, $r_\mu = r$ and $\phi_\mu = \phi$, or or $t_\mu = t$, $\theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$.*

We consider three subcases following.

Subcase 3.1. *$t_\mu = t$, $r_\mu = r$ and $\theta_\mu = \theta$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)^{-1} dr^2 - mr^2d\theta^2 - r^2\sin^2\theta\sum_{\mu=1}^{m}d\phi_\mu^2.$$

Subcase 3.2. *$t_\mu = t$, $r_\mu = r$ and $\phi_\mu = \phi$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)^{-1} dr^2 - r^2\sum_{\mu=1}^{m}(d\theta_\mu^2 + \sin^2\theta_\mu d\phi^2).$$

There subcases 3.1 and 3.2 can be only happen if the centers of these m fields are at a same point O in a space.

Subcase 3.3. *$t_\mu = t$, $\theta_\mu = \theta$ and $\phi_\mu = \phi$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r_\mu}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r_\mu}\right)^{-1} dr_\mu^2 - \sum_{\mu=1}^{m}r_\mu(d\theta^2 + \sin^2\theta d\phi^2).$$

Case 4. *$\widehat{m} = 4$, i.e., $t_\mu = t, r_\mu = r, \theta_\mu = \theta$ and $\phi_\mu = \phi$ for $1 \leq \mu \leq m$.*

In this subcase, the combinatorial metric is

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)dt^2 - \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2r}\right)^{-1} dr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Particularly, if $m_\mu = M$ for $1 \leq \mu \leq m$, we get that

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)mdt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1} mdr^2 - mr^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Define a coordinate transformation $(t, r, \theta, \phi) \rightarrow ({}_st, {}_sr, {}_s\theta, {}_s\phi) = (t\sqrt{m}, r\sqrt{m}, \theta, \phi)$. Then the previous formula turns to

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)d_st^2 - \frac{d_sr^2}{1 - \frac{2GM}{c^2r}} - {}_sr^2(d_s\theta^2 + \sin^2{}_s\theta d_s\phi^2)$$

in this new coordinate system ($_st,\ _sr,\ _s\theta,\ _s\phi$), whose geometrical behavior likes that of the gravitational field.

Consider $\widehat{m}$ the discussion is divided into two cases, which lead to two opposite conclusions following.

Case 1. $\widehat{m} = 4$.

In this case, we get that $\dim\widetilde{R} = 3$, i.e., all Euclidean spaces $\mathbf{R}_1^3, \mathbf{R}_2^3, \cdots, \mathbf{R}_m^3$ are in one $\mathbf{R}^3$, which is the most enjoyed case by human beings. If it is so, all the behavior of Universe can be realized finally by human beings, particularly, the observed interval is ds and all natural things can be come true by experiments. This also means that the discover of science will be ended, i.e., we can find an ultimate theory for our cosmos - the *Theory of Everything*. This is the earnest wish of Einstein himself beginning, and then more physicists devoted all their lifetime to do so in last century.

Case 2. $\widehat{m} \leq 3$.

If our cosmos is so, then $\dim\widetilde{R} \geq 4$. In this case, the observed interval in the field $\mathbf{R}_{human}^3$ where human beings live is

$$ds_{human}^2 = a(t,r,\theta,\phi)dt^2 - b(t,r,\theta,\phi)dr^2 - c(t,r,\theta,\phi)d\theta^2 - d(t,r,\theta,\phi)d\phi^2.$$

by Schwarzschild metrics in R^3. But we know the metric in $\widetilde{R}$ should be $ds_{\widetilde{R}}$. Then

how to we explain the differences $(ds_{\widetilde{R}} - ds_{human})$ *in physics?*

Notice that one can only observes the line element ds_{human}, i.e.,, a projection of $ds_{\widetilde{R}}$ on $\mathbf{R}_{human}^3$ by the projective principle. Whence, all contributions in $(ds_{\widetilde{R}} - ds_{human})$ come from the spatial direction not observable by human beings. In this case, it is difficult to determine the exact behavior and sometimes only partial information of the Universe, which means that each law on the Universe determined by human beings is an approximate result and hold with conditions.

Furthermore, if $\widehat{m} \leq 3$ holds, because there are infinite underlying connected graphs, i.e., there are infinite combinations of $\mathbf{R}^3$, one can not find an ultimate theory for the Universe, which means the discover of science for human beings will endless forever, i.e., there are no a *Theory of Everything*.

§10.4 A COMBINATORIAL MODEL FOR CIRCULATING ECONOMY

10.4.1 Input-Output Analysis in Macro-Economy. Assume these are n departments $D_1, D_2, \cdots, D_n$ in a macro-economic system $\mathscr{L}$ satisfy conditions following:

(1) The total output value of department D_i is x_i. Among them, there are x_{ij} output values for the department D_j and d_i for the social demand, such as those shown in Fig.10.4.1.

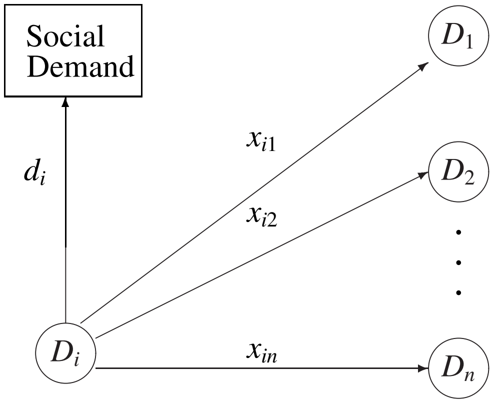


Fig.10.4.1

(2) A unit output value of department D_j consumes t_{ij} input values coming from department D_i. Such numbers t_{ij}, $1 \leq i, j \leq n$ are called *consuming coefficients*.

Therefore, such a overall balance macro-economic system $\mathscr{L}$ satisfies n linear equations

$$x_i = \sum_{j=1}^{n} x_{ij} + d_i \qquad (10-1)$$

for integers $1 \leq i \leq n$. Furthermore, substitute $t_{ij} = x_{ij}/x_j$ into equation (10-1), we get that

$$x_i = \sum_{j=1}^{n} t_{ij} x_j + d_i \qquad (10-2)$$

for any integer i. Let $\mathbf{T} = [t_{ij}]_{n\times n}$, $\mathbf{A} = I_{n\times n} - T$. Then

$$\mathbf{A}\overline{x} = \overline{d}, \qquad (10-3)$$

from (10-2), where $\overline{x} = (x_1, x_2, \cdots, x_n)^T$, $\overline{d} = (d_1, d_2, \cdots, d_n)^T$ are the output vector or demand vectors, respectively. This is the famous *input-output model* in macro-economic analysis established by a economist Leontief won the Nobel economic prize in 1973.

For an simple example, let $\mathscr{L}$ consists of 3 departments D_1, D_2, D_3, where D_1=agriculture, D_2=manufacture industry, D_3=service with an input-output data in Table 10.4.1([TaC1]).

Department	D_1	D_2	D_3	Social demand	Total value
D_1	15	20	30	35	100
D_2	30	10	45	115	200
D_3	20	60	/	70	150

Table 10.4.1

This table can be turned to a consuming coefficient table by $t_{ij} = x_{ij}/x_j$ following.

Department	D_1	D_2	D_3
D_1	0.15	0.10	0.20
D_2	0.30	0.05	0.30
D_3	0.20	0.30	0.00

Thus

$$\mathbf{T} = \begin{bmatrix} 0.15 & 0.10 & 0.20 \\ 0.30 & 0.05 & 0.30 \\ 0.20 & 0.30 & 0.00 \end{bmatrix}, \quad \mathbf{A} = I_{3\times 3} - \mathbf{T} = \begin{bmatrix} 0.85 & -0.10 & -0.20 \\ -0.30 & 0.95 & -0.30 \\ -0.20 & -0.30 & 1.00 \end{bmatrix}$$

and the input-output equation system is

$$\begin{aligned} 0.85x_1 - 0.10x_2 - 0.20x_3 &= d_1 \\ -0.30x_1 + 0.95x_2 - 0.30x_3 &= d_2 \\ -0.20x_1 - 0.30x_2 + x - 3 &= d_3 \end{aligned}$$

Solving this equation system enables one to find the input and output data for economy.

Notice that the WORLD is not linear in general, i.e., the assumption $t_{ij} = x_{ij}/x_j$ does not hold in general. A non-linear input-output model is shown in Fig.10.4.2, where $\overline{x} = (x_{1i}, x_{2i}, \cdots, x_{ni})$, $D_1, D_2, \cdots, D_n$ are n departments and SD=social demand. Usually, the function $F(\overline{x})$ is called the *producing function*.

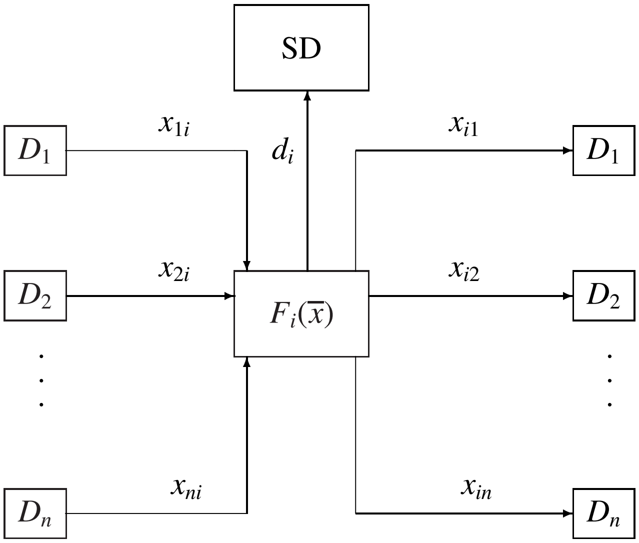


Fig.10.4.2

Thus a general overall balance input-output model is characterized by equations

$$F_i(\overline{x}) = \sum_{j=1}^{n} x_{ij} + d_i, \qquad (10-4)$$

for integers $1 \leq i \leq n$, where $F_i(\overline{x})$ may be linear or non-linear.

10.4.2 Circulating Economic System. A scientific economical system should be a conservation system of human being with nature in harmony, i.e., to make use of matter and energy rationally and everlastingly, to decrease the unfavorable effect that economic activities may make upon our natural environment as far as possible, which implies to establish a circulating economic system shown in Fig.10.4.3.

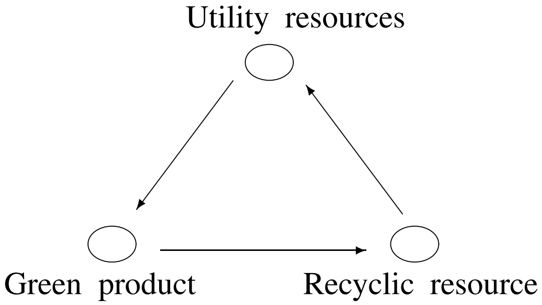


Fig.10.4.3

Generally, a *circulating economic system* is such a overall balance input-output multi-space $\widetilde{M} = \bigcup\limits_{i=1}^{k} M_i(t)$ that there are no rubbish in each producing department. Particularly, there are no harmful wastes to the WORLD. Thus any producing department of a circulating economical system is in a locally economical system $M_1(t), M_2(t), \cdots, M_k(t)$ underlying a directed circuit $G\left[\widetilde{M}_L\right] = \overrightarrow{C}_k$ for an integer $k \geq 2$, such as those shown in Fig.10.4.4.

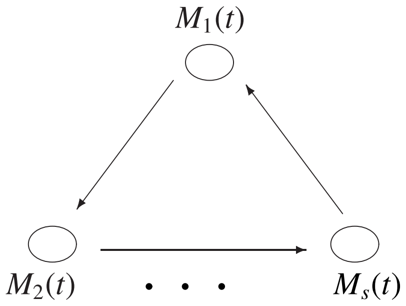


Fig.10.4.4

Consequently, we get a structure result for circulating economic system following.

Theorem 10.4.1 *Let $\widetilde{M}(t)$ be a circulating economic system consisting of producing departments $M_1(t)$, $M_2(t), \cdots, M_n(t)$ underlying a graph $G\left[\widetilde{M}(t)\right]$. Then there is a circuit-decomposition*

$$G\left[\widetilde{M}(t)\right] = \bigcup_{i=1}^{l} \overrightarrow{C}_s$$

for the directed graph $G\left[\widetilde{M}(t)\right]$ such that each output of a producing department $M_i(t)$, $1 \leq i \leq n$ is on a directed circuit $\overrightarrow{C}_s$ for an integer $1 \leq s \leq l$.

Similarly, assume that there are n producing departments $M_1(t)$, $M_2(t), \cdots, M_n(t)$, x_{ij} output values of $M_i(t)$ for the department $M_j(t)$ and d_i for the social demand. Let $F_i(x_{1i}, x_{2i}, \cdots, x_{ni})$ be the producing function in $M_i(t)$. Then a circulating economic system can be characterized by equations

$$F_i(\overline{x}) = \sum_{j=1}^{n} x_{ij} + d_i, \qquad (10-5)$$

for integers $1 \leq i \leq n$ with each x_{ij} on one and only one directed circuit consisting some of departments $M_1(t)$, $M_2(t), \cdots, M_n(t)$, such as those shown in Fig.10.4.4.

§10.5 A COMBINATORIAL MODEL FOR CONTAGION

10.5.1 Infective Model in One Space. Let N be the number of persons in considered group $\mathscr{G}$. Assume that there are only two kind groups in persons at time t. One is infected crowd. Another is susceptible crowd. Denoted by $I(t)$ and $S(t)$, respectively. Thus $S(t) + I(t) = N$, i.e., $(S(t)/N) + (I(t)/N) = 1$. The numbers $S(t)/N$, $I(t)/N$ are called susceptibility or infection rate and denoted by $S(t)$ and $I(t)$ usually, i.e., $S(t) + I(t) = 1$. If N is sufficiently large, we can further assume that $S(t)$, $I(t)$ are smoothly.

Assume that the infected crowd is a direct proportion of susceptible crowd. Let k be such a rate. Thus an infected person can infects $kS(t)$ susceptible persons. It is easily know that there are $N(I(t + \Delta t) - I(t))$ new infected persons in the time interval $[t, t + \Delta t]$. We know that

$$N(I(t + \Delta t) - I(t)) = kNS(t)I(t)\Delta t.$$

Divide its both sides by $N\Delta t$ and let $t \to \infty$, we get that

$$\frac{dI}{dt} = kIS.$$

Notice that $S(t) + I(t) = 1$. We finally get that

$$\begin{cases} \dfrac{dI}{dt} = kI(1 - I), \\ I(0) = I_0. \end{cases} \qquad (10-6)$$

This is the *SI model of infectious disease* on infected diseases. Separating variables we get that

$$I(t) = \frac{1}{1 + (I_0^{-1} - 1)} e^{-kt}$$

and

$$S(t) = 1 - I(t) = \frac{(I_0^{-1} - 1)e^{-kt}}{1 + (I_0^{-1} - 1)e^{-kt}}.$$

Clearly, if $t \to +\infty$, then $I(t) \to 1$ in SI model. This is not in keeping with the actual situation. Assume the rate of heal persons in infected persons is h. Then $1/h$ denoted the infective stage of disease. The SI model (10-6) is reformed to

$$\begin{cases} \dfrac{dI}{dt} = kI(1 - I) - hI, \\ I(0) = I_0, \end{cases} \qquad (10-7)$$

called the *SIS model of infectious disease*. Similarly, by separating variables we know that

$$I(t) = \begin{cases} \left[e^{-(k-h)t}\left(\frac{1}{I_0} - \frac{1}{1-\sigma^{-1}}\right) + \frac{1}{1-\sigma^{-1}} \right]^{-1}, \\ \frac{I_0}{ktI_0 + 1}, \end{cases}$$

where $\sigma = k/h$ is the number of average infections by one infected person in an infective stage. Clearly,

$$\lim_{t\to+\infty} I(t) = \begin{cases} 1 - \frac{1}{\sigma} & \text{if } \sigma > 1 \\ 0 & \text{if } \sigma \leq 1. \end{cases}$$

Consequently, if $\sigma \leq 1$, the infection rate is gradually little by little, and finally approaches 0. But if $\sigma \geq 1$, the increase or decrease of $I(t)$ is dependent on I_0. In fact, if $I_0 < 1 - \sigma^{-1}$, $I(t)$ is increasing and it is decreasing if $I_0 > 1 - \sigma^{-1}$. Both of them will let $I(t)$ tend to a non-zero limitation $1 - \sigma^{-1}$. Thus we have not a radical cure of this disease.

Now assume the heal persons acquired immunity after infected the decrease and will never be infected again. Denoted the rate of such persons by $R(t)$. Then $S(t)+I(t)+R(t) = 1$ and the SIS model (10-7) is reformed to

$$\begin{cases} \frac{dS}{dt} = -kIS, \\ \frac{dI}{dt} = kIS - hI, \\ S(0) = S_0, I(0) = I_0, R(0) = 0, \end{cases} \qquad (10-8)$$

called the *SIR model of infectious disease*. These differential equations are first order non-linear equations. We can not get the analytic solutions $S(t)$, $I(t)$.

Furthermore, let I and J be respectively diagnosis of infection and non-diagnosis infection. Let k_1, k_2 be the infection rate by an infection, or a diagnosis, h_1, h_2 the heal rate from infection or diagnosis and the detecting rate of this infectious disease by α. We get the following *SIJR model of infectious disease*

$$\begin{cases} \frac{dS}{dt} = -(k_1 I + k_2 J)S, \\ \frac{dI}{dt} = (k_1 I + k_2 J)S - (\alpha + h_1)I, \\ \frac{dJ}{dt} = \alpha I - h_2 J, \\ \frac{dR}{dt} = h_1 I + h_2 J, \\ S(0) = S_0,\ I(0) = I_0, \\ J(0) = J_0,\ R(0) = R_0, \end{cases} \qquad (10-9)$$

which are also first order differential non-linear equations and can be find behaviors by qualitative analysis only.

10.5.2 Combinatorial Model on Infectious Disease. Let $C_1, C_2, \cdots, C_m$ be m segregation crowds, i.e., a person moving from crowds C_i to C_j can be only dependent on traffic means with persons $N_1, N_2, \cdots, N_m$, respectively. For an infectious disease, we assume that there are only two kind groups in C_i, namely the infected crowd $I_i(t)$ and susceptible crowd $S_i(t)$ for integers $1 \leq i \leq m$. Among them, there are $U_i(t)$, $V_i(t)$ persons moving in or away C_i at time t. Thus $S_i(t) + I_i(t) - U_i(t) + V_i(t) = N_i$. Denoted by $c_{ij}(t)$ the persons moving from C_i to C_j for integers $1 \leq i, j \leq m$. Then

$$\sum_{s=1}^{m} c_{si}(t) = U_i(t) \quad \text{and} \quad \sum_{s=1}^{m} c_{is}(t) = V_i(t).$$

A combinatorial model of infectious disease is defined by labeling graph G^L following:

$$V\left(G^l\right) = \{C_1, C_2, \cdots, C_m\},$$
$$E\left(G^l\right) = \{(C_i, C_j) |\ \text{there are traffic means from } \mathrm{C_i} \text{ to } \mathrm{C_j},\ 1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{m}\ \};$$
$$l(C_i) = N_i, \quad l^+(C_i, C_j) = c_{ij}$$

for $\forall (C_i, C_j) \in E\left(G^l\right)$ and integers $1 \leq i, j \leq m$. Such as those shown in Fig.10.5.1.

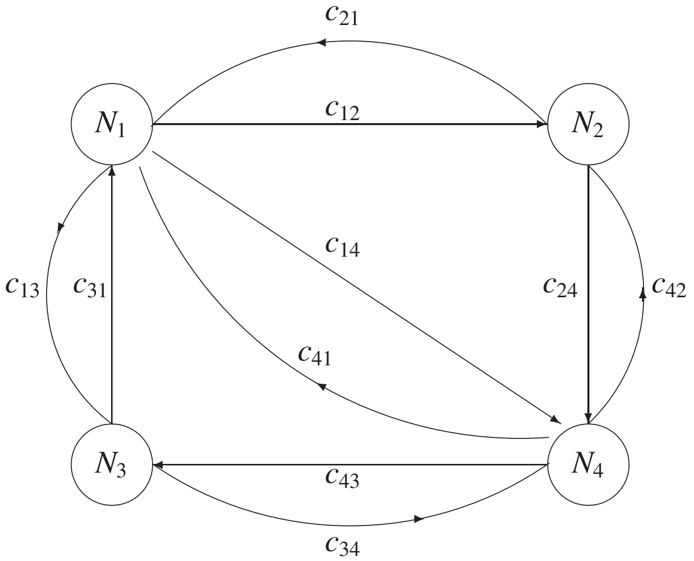


Fig.10.5.1

Similarly, assume that an infected person can infects k susceptible persons and $c_{ij} = t_{ij}N_i$, where t_{ij} is a constant. Then the number of persons in crowd C_i is also a constant

$$N_i\left(1+\sum_{s=1}^{n}(t_{si}-t_{is})\right).$$

In this case, the SI, SIS, SIR and SIJR models of infectious disease for crowd C_i turn to

$$\begin{cases}\frac{dI_i}{dt}=kI_i(1-I_i),\\ I_i(0)=I_{i0}.\end{cases}\qquad(10-10)$$

$$\begin{cases}\frac{dI_i}{dt}=kI_i(1-I_i)-hI_i,\\ I_i(0)=I_{i0},\end{cases}\qquad(10-11)$$

$$\begin{cases}\frac{dS_i}{dt}=-kI_iS_i,\\ \frac{dI_i}{dt}=kI_iS_i-hI_i,\\ S_i(0)=S_{i0},I_i(0)=I_{i0},R(0)=0,\end{cases}\qquad(10-12)$$

$$\begin{cases}\frac{dS_i}{dt}=-(k_1I_i+k_2J_i)S_i,\\ \frac{dI_i}{dt}=(k_1I_i+k_2J)S-(\alpha+h_1)I,\\ \frac{dJ_i}{dt}=\alpha I_i-h_2J_i,\\ \frac{dR_i}{dt}=h_1I_i+h_2J_i,\\ S_i(0)=S_{i0},\ I_i(0)=I_{i0},\\ J_i(0)=J_{i0},\ R_i(0)=R_{i0}\end{cases}\qquad(10-13)$$

if there are always exist a contagium in C_i for any integer $1\leq i\leq m$, where h and R are the respective rates of heal persons in infected persons and the heal persons acquired immunity after infected, k_1, k_2 the infection rate by an infection, or a diagnosis, h_1, h_2 the heal rate from infection or diagnosis and α the detecting rate of the infectious disease. Similarly, we can solve SI or SIS models by separating variables. For example,

$$I_i(t)=\begin{cases}\left[e^{-(k-h)t}\left(\frac{1}{I_0}-\frac{1}{1-\sigma^{-1}}\right)+\frac{1}{1-\sigma^{-1}}\right]^{-1},\\ \frac{I_0}{ktI_0+1}\end{cases}$$

for the SIS model of infectious disease, where $\sigma=k/h$. Thus we can control the infectious likewise that in one space.

But the first contagium can only appears in one crowd, for instance C_1. As we know, the purpose of infectious disease is to know well its infection rule, decrease or eliminate such disease. Applying the combinatorial model of infectious disease, an even more effective measure is isolating contagia unless to cure the infections, which means that we need to cut off all traffic lines from the contagia appeared crowds, for example, all such traffic lines (C_1, C_s), (C_s, C_1) for integers $1 \leq s \leq m$, where C_1 is the crowd found the first contagium.

§10.6 RESEARCH PROBLEMS

10.6.1 In fact, Smarandache multi-space is a systematic notion on objectives. More and more its applications to natural sciences and humanities are found today. The readers are refereed to [Mao37]-[Mao38] for its further applications, and also encouraged to apply it to new fields or questions.

10.6.2 The combinatorial model on cosmos presents research problems to both physics and combinatorics, such as those of the following:

Problem 10.6.1 *Embed a connected graph into Euclidean spaces of dimension*≥ 4*, research its phase space and apply it to cosmos.*

Motivated by this combinatorial model on cosmos, a number of conjectures on cosmoses are proposed following.

Conjecture 10.6.1 *There are infinite many cosmoses and all dimensions of cosmoses make up an integer interval* $[1, +\infty]$.

A famous proverbs in Chinese says that *seeing is believing but hearing is unbelieving*, which is also a dogma in the pragmatism. Today, this view should be abandoned for a scientist if he wish to understand the WORLD. On the first, we present a conjecture on the traveling problem between cosmoses.

Conjecture 10.6.2 *There exists beings who can get from one cosmos into another, and there exists being who can enter the subspace of lower dimension from that of higher dimensional space, particularly, on the earth.*

Although nearly every physicist acknowledges the existence of black and white holes. All these holes are worked out by mathematical calculation, not observation of human beings.

Conjecture 10.6.3 *A black hole is also a white hole in space, different from the observation in or out the observed space.*

Our cosmonauts is good luck if Conjecture 6.6.3 holds since they are never needed for worrying about attracted by these black holes. Today, an important task for experimental physicists is looking for dark matters on the earth. However, this would be never success by the combinatorial model of cosmos, included in the next conjecture.

Conjecture 10.6.4 *One can not find dark matters by experiments on the earth because they are in spatial can not be found by human beings.*

Few consideration is on the relation of dark energy with dark matters. We believe that there exists relations between them, particularly, the following conjecture.

Conjecture 10.6.5 *The dark energy is nothing but a kind of effect of internal action in dark matters and the action of black on white matters. One can only surveys the acting effect of black matters on that of white, will never be the all.*

10.6.3 The input-output model is a useful in macro-economy analysis. In fact, any system established by scientist is such an input-output system by combinatorial speculation. Certainly, these systems are non-linear in general because our WORLD is non-linear.

Problem 10.6.2 *Let $F_i(\overline{x})$ be a polynomial of degree $d \geq 2$ for integers $1 \leq i \leq n$. Solve equations (10-5) for circulating economic system underlying a graph G, particularly, $d = 2$ and $G \simeq C_n$ or $n \leq 4$.*

Problem 10.6.3 *Let $F_i(\overline{x})$ be a well-known functions $f(x)$, such as those of $f(x) = x^{\mu}$, where μ is a rational number, or $f(x) = \ln x, \ \sin x, \ \cos x, \ \cdots$, etc.. Determine such conditions that equations (10-5) are solvable.*

10.6.4 We have shown in Subsection 10.5.2 that one can control an infectious disease in a combinatorial space likewise that in one space if assume that the number c_{ij} of persons moving from crowd C_i to C_j is a proportion of persons in crowd C_i, i.e., $c_{ij} = t_{ij}N_i$ with a constant t_{ij} for integers $1 \leq i, j \leq m$. Such a assumption is too special, can not hold in general.

Problem 10.6.4 *Establish SI, SIS, SIR and SIJR models of infectious disease without the* *assumption* $c_{ij} = t_{ij}N_i$, *i.e.,* $c_{ij} = f(N_i, N_j, t)$ *for integers* $1 \leq i, j \leq m$ *and solve these differential equations.*

Although these differential equations maybe very different from these equations (10-10)-(10-13), the measure by isolating contagia and cutting off all traffic lines from the contagia appeared crowds is still effective for control a disease in its infective stage. That is why this measure is usually adopted when an infective disease occurs.

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